X. *On the Mathematical Theory of Stream-lines, especially those with four Foci and upwards.* By WILLIAM JOHN MACQUORN RANKINE, *C.E., LL.D., F.R.SS. Lond. & Edin., &c.*

Received January 1,—Read February 10, 1870.

Introduction.

§ 1. *Object and Occasion of this Investigation.—A Stream-line* is the line that is traced by a particle in a steady current of fluid. Each individual stream-line preserves its figure and position unchanged, and marks the track of a filament or continuous series of particles that follow each other. The motions in different parts of a steady current may be represented to the eye and to the mind by means of a group of stream-lines; for the direction of motion of a particle at a given point is that of a tangent to the stream-line which traverses that point; and when the fluid is of constant density, as is sensibly the case with liquids, the comparative velocities at different points are indicated by the comparative closeness of the stream-lines to each other. Even when the fluid is gaseous, the comparative *mass-velocities* are indicated by the closeness of the stream-lines—the term *mass-velocity* meaning the mass which traverses a unit of area in a unit of time. Gaseous fluids, however, will not be considered in the present paper.

Stream-lines are important in connexion with naval architecture; for the curves which the particles of water describe relatively to a ship, in moving past her, are stream-lines; and if the figure of a ship is such that the particles of water glide smoothly over her skin, that figure is a *stream-line surface**, being a surface which contains an indefinite number of stream-lines. The stream-lines of a current gliding past a circular cylinder in a direction transverse to its axis, and also those of a current gliding past a sphere, have long been known.

In a paper entitled "On Plane Water-lines in two Dimensions," read to the Royal Society in 1863, and published in the Philosophical Transactions, I have given a detailed

* Note added December 1870.—This limitation is necessary in speaking of the figures of ships; for although every surface is a possible stream-line surface, the surface of a ship is not even approximately an actual stream-line surface unless it is such that she does not drag along with her a mass of eddies of such volume and shape as to cause the actual tracks of the particles of water to differ materially in form from those which would be described in the absence of eddies. The surfaces which fulfil this condition are what are called by shipbuilders "*fair*" surfaces; and their forms have in a great many cases been determined by practical experience. In order to determine, at all events approximately, the actions of such surfaces on the water, it is necessary to be able to construct them by geometrical rules based on the principles of the motion of fluids; and the methods described in this paper afford the means of doing so.—W. J. M. R.

investigation of the mathematical properties of a very extensive class of stream-lines, representing the motions of particles of water in layers of uniform thickness. Those stream-lines closely resemble the water-lines, riband-lines, and other longitudinal sections of ships of a great variety of forms and proportions; and there is scarcely any known figure of a fair longitudinal line on a ship's skin to which an approximation may not be found amongst them; hence I have proposed to call them Neoïds; that is, ship-shape lines.

In the Philosophical Magazine for October 1864, was published a paper which had been read by me to the British Association, containing a summary of the properties of some additional kinds of stream-lines, some in two, and others in three dimensions, and of those stream-lines in particular which generate stream-line surfaces of revolution. All these stream-lines also are neoïds, or ship-shape curves.

All the neoïd stream-lines before mentioned are either *unifocal* or *bifocal*; that is to say, they may be conceived to be generated by the combination of a uniform progressive motion with another motion consisting in a divergence of the particles from a certain point or focus, followed by a convergence either towards the same point or towards a second point. Those which are continuous closed curves, when unifocal are circular, and when bifocal are blunt-ended ovals, in which the length may exceed the breadth in any given proportion—for example, the curves marked L B in figs. 2, 3 & 4, Plate XV. To obtain a unifocal or bifocal neoïd resembling a longitudinal line of a ship with sharp ends, such as A, fig. 1, it is necessary to take a part only of a stream-line, and then there is discontinuity of form and of motion at each of the two ends of that line.

The occasion of the investigation described in the present paper was the communication to me by Mr. WILLIAM FROUDE of some results of experiments of his on the resistance of model boats, of lengths ranging from 3 to 12 feet. A summary of those results is published at the end of a Report to the British Association, "On the State of Existing Knowledge of the Qualities of Ships." In each case two models were compared together of equal displacement and equal length; the water-line of one was a wave-line, as at A (Plate XV. fig. 1), with fine sharp ends; that of the other had blunt rounded ends, as at B—suggested, Mr. FROUDE states, by the appearance of water-birds when swimming. At low velocities, the resistance of the sharp-ended boat was the smaller; at a certain velocity, bearing a definite relation to the length of the model, the resistances became equal; and at higher velocities the round-ended model had a rapidly increasing advantage over the sharp-ended model.

Hence it appeared to me to be desirable to investigate the mathematical properties of stream-lines resembling the water-lines of Mr. FROUDE'S bird-like models; and I have found that endless varieties of such forms, all closed curves free from discontinuity of form and of motion, may be obtained by using four foci instead of two. They may be called, from this property, *quadrifocal stream-lines*, or, from the idea that suggested such shapes to Mr. FROUDE, *Cycnoïds,* or swan-like lines; while the stream-lines in which particles of liquid flow past them may be said to be *Cycnogenous**.

CHAPTER I. *Summary of Cinematical Principles.*

§ 2. *Normal Surfaces to Stream-lines in a Liquid.* (For details on this part of the subject, see STOKES "On the steady Motion of an Incompressible Fluid," Cambridge Transactions, 1842; also RANKINE "On Plane Water-lines in Two Dimensions," Philosophical Transactions, 1863.)—Let a perfectly liquid mass of indefinite extent flow past a solid body in such a manner that, as the distance from the solid body in any direction increases without limit, the motion of the liquid particles approaches indefinitely to uniformity in velocity and direction. Let u, v, and w be the rectangular components of the velocity of any particle; then the condition of constant density requires that the following equation should be fulfilled,

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0; \quad \ldots \ldots \quad (1)$$

and the condition of perfect fluidity being combined with that of the approximation to uniformity of motion at an indefinite distance requires that the three following equations should be fulfilled:

$$\frac{dv}{dz}-\frac{dw}{dy}=0;\ \frac{dw}{dx}-\frac{du}{dz}=0;\ \frac{du}{dy}-\frac{dv}{dx}=0. \quad \ldots \ldots \quad (2)$$

These four conditions are fulfilled by making

$$u=\frac{d\phi}{dx};\ v=\frac{d\phi}{dy};\ w=\frac{d\phi}{dz}; \quad \ldots \ldots \quad (3)$$

the *velocity-function*, ϕ, being a function which fulfils the condition

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\phi=0. \quad \ldots \ldots \quad (4)$$

The equation

$$\phi=a \text{ (a constant)} \quad \ldots \ldots \quad (5)$$

is that of a surface of equal action, which is normal to the direction of motion of every particle that it traverses; in other words, it is normal to all the stream-lines that it cuts. If a series of different values be given to the constant a, the equation (5) represents a series of such normal surfaces; and every stream-line is a normal trajectory to that series of surfaces. In symbols, let ds' denote an elementary arc of a stream-line, and x', y', and z' the coordinates of a fixed point in it, those coordinates being regarded as functions

* Κυκνοειδής, κυκνογενής. It is to be observed that the swan-like curves here described are different from the lines of the vessel which some years ago was built from the designs of Mr. PEACOCK, and described in the Mechanics' Magazine; for the lines of that vessel are oval, and approximate to bifocal neoïds, and are wholly without the peculiarly shaped ends that characterize Mr. FROUDE's cycnoïd models.

of s'; then we have

$$\left.\frac{\frac{dx'}{ds'}}{\frac{d\varphi}{dx}}=\frac{\frac{dy'}{ds'}}{\frac{d\varphi}{dy}}=\frac{\frac{dz'}{ds'}}{\frac{d\varphi}{dz}}.\right\} \quad \ldots \ldots \ldots \quad (6)$$

In short, the stream-lines bear the same relation to the normal surfaces that lines of force bear to equipotential surfaces.

Let the axis of x be taken parallel to the direction of the uniform motion of the particles at an indefinitely great distance from the origin of coordinates, near to which the solid body is supposed to be situated; and let the velocity of that uniform current be taken as the unit of velocity, so that u, v, and w shall represent the ratios of the three components of the velocity of a particle to the velocity at an indefinite distance. Then, when either x, y, or z is indefinitely great, we have

$$u=1;\ v=0;\ w=0;$$

and it is evident that the velocity-function must be of the following form,

$$\varphi=x+\varphi_1, \quad \ldots \ldots \ldots \quad (7)$$

in which φ_1 is a function that vanishes when x, y, or z increases indefinitely. The term x gives, by its differentiations, the expression of a uniform straight current, of the velocity 1. The term φ_1 gives, by its differentiations, the three components of the *disturbance* of the velocity from that of the uniform current. Hence, if we suppose the water at an indefinite distance from the disturbing solid to be still, and the solid to move parallel to the axis of x with the velocity -1, the following coefficients,

$$\frac{d\varphi_1}{dx},\ \frac{d\varphi_1}{dy},\ \frac{d\varphi_1}{dz},$$

will represent the components of the velocity of a particle *relatively to still water.*

§ 3. *Stream-line Surfaces in general.*—For some purposes a more convenient way of expressing the properties of stream-lines is, to consider the system of stream-lines in a steadily moving current of liquid as the intersections of two sets of surfaces called *stream-line surfaces*, represented by the two sets of equations

$$\psi=b;\ \chi=c, \quad \ldots \ldots \ldots \quad (8)$$

where b and c are constants, each of which receives a series of different values. Each set of surfaces divides the space in which the current flows into a series of indefinitely thin layers; and the two sets of surfaces divide that space into a series of indefinitely slender *elementary streams**, which are conceived to be of *equal flow*. The uniform current at an indefinite distance from the disturbing solid being, as before, parallel to x, and of the velocity 1, let the transverse area of an elementary stream at an indefinite distance be denoted by σ; the same symbol denotes the volume of the flow in each unit of time along that stream, and therefore along every elementary stream. The areas of

* Note added June 1871.—Called by CLERK MAXWELL "unit-tubes."

the three sections of an elementary stream, made at a given point by three planes parallel to the three coordinate planes respectively, have the following values:

$$\text{parallel to } yz, =\frac{\sigma}{\frac{d\psi}{dy}\cdot\frac{d\chi}{dz}-\frac{d\psi}{dz}\cdot\frac{d\chi}{dy}};$$

and symmetrical expressions for those parallel to zx and to xy respectively.

The three components of the velocity of an elementary stream at a given point are to be found by dividing the volume of flow by the areas of those three sections respectively; hence those components are as follows:—

$$u=\frac{d\psi}{dy}\cdot\frac{d\chi}{dz}-\frac{d\psi}{dz}\cdot\frac{d\chi}{dy}=\frac{d\phi}{dx} \quad \ldots \quad (9)$$

(and symmetrical expressions for v and w).

The third member of the equation is introduced in order to show the relations between the stream-line functions ψ and χ, and the velocity-function ϕ.

It is easily ascertained that the preceding values of u, v, and w fulfil the condition of constant density (equation 1); also that the surfaces of equal action ($\phi=a$) cut the stream-line surfaces at right angles, as expressed by the following equations:

$$\left.\begin{aligned}\frac{d\psi}{dx}\cdot\frac{d\phi}{dx}+\frac{d\psi}{dy}\cdot\frac{d\phi}{dy}+\frac{d\psi}{dz}\cdot\frac{d\phi}{dz}=0;\\ \frac{d\chi}{dx}\cdot\frac{d\phi}{dx}+\frac{d\chi}{dy}\cdot\frac{d\phi}{dy}+\frac{d\chi}{dz}\cdot\frac{d\phi}{dz}=0.\end{aligned}\right\} \quad \ldots \quad (10)$$

The conditions expressed by the three equations (2) take in the present instance the following form:

$$\left.\begin{aligned}0&=\frac{dv}{dz}-\frac{dw}{dy}\\ &=-\frac{d\psi}{dx}\left(\frac{d^2\chi}{dy^2}+\frac{d^2\chi}{dz^2}\right)+\frac{d\psi}{dy}\cdot\frac{d^2\chi}{dxdy}+\frac{d\psi}{dz}\cdot\frac{d^2\chi}{dzdx}\\ &\quad+\frac{d\chi}{dx}\left(\frac{d^2\psi}{dy^2}+\frac{d^2\psi}{dz^2}\right)-\frac{d\chi}{dy}\cdot\frac{d^2\psi}{dxdy}-\frac{d\chi}{dz}\cdot\frac{d^2\psi}{dzdx};\\ 0&=\frac{dw}{dx}-\frac{du}{dz}=\text{(expression formed by symmetry)};\\ 0&=\frac{du}{dy}-\frac{dv}{dx}=\text{(expression formed by symmetry)}.\end{aligned}\right\} \quad \ldots \quad (11)$$

The preceding set of three equations show the whole conditions which the functions ψ and χ must fulfil, in order that they may represent stream-line surfaces.

In finding the point in a stream line where a given function F is a maximum, the condition to be fulfilled is

$$\frac{dF}{dt}=\left(u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}\right)F=0. \quad \ldots \quad (11\text{ A})$$

The following formula is an immediate consequence of the equations (2): let dx' denote an elementary line in any direction, and u' the component velocity of a particle along dx'; then

$$\frac{du'}{dt}=\frac{d}{dx'}\left(\frac{u^2+v^2+w^2}{2}\right). \quad \text{(11 B)}$$

In the two previous papers before referred to, a class of stream-lines is described under the name of *Lissoneoïds*, whose characteristic property is that two maxima and one minimum of the velocity coalesce in one point, at the greatest breadth of the figure bounded by the line. The mathematical properties of a lissoneoïd are expressed by the following set of equations:

$$\left.\begin{array}{l} \text{when } x=0;\ \text{let } v=0;\ w=0; \\ \frac{d}{dt}(u^2+v^2+w^2)=0; \\ \frac{d^2}{dt^2}(u^2+v^2+w^2)=0; \end{array}\right\} \quad \text{(11 C)}$$

and it can be shown that for the last two of these equations the following may be substituted in the cases which occur in practice:

$$u\frac{d^2u}{dx^2}+2\frac{du^2}{dy^2}+2\frac{du^2}{dz^2}=0. \quad \text{(11 D)}$$

In order to express the condition that at an indefinitely great distance from the origin v and w shall vanish, and u approximate indefinitely to 1, it is necessary that, when either x, y, or z increases indefinitely, the functions ψ and χ shall approximate indefinitely to two functions of y and z only, which may be denoted by ψ_0 and χ_0, fulfilling the following conditions,

$$\left.\begin{array}{l} \frac{d\psi_0}{dy}\frac{d\chi_0}{dz}-\frac{d\psi_0}{dz}\frac{d\chi_0}{dy}=1; \\ \frac{d\psi_0}{dx}=0;\ \frac{d\chi_0}{dx}=0; \end{array}\right\} \quad \text{(12)}$$

that is to say, first, the surfaces represented by ψ_0 and χ_0 divide the space into elementary streams of equal transverse area; secondly, these surfaces are plane or cylindrical, and parallel to the axis of x; and thirdly, they are asymptotic to the surfaces represented by ψ and χ. Let us now make

$$\psi=\psi_0+\psi_1;\ \chi=\chi_0+\chi_1; \quad \text{(13)}$$

then the equations (9) take the following form:

$$\left.\begin{array}{l} u=1+\frac{d\psi_0}{dy}\cdot\frac{d\chi_1}{dz}+\frac{d\psi_1}{dy}\cdot\frac{d\chi_0}{dz}+\frac{d\psi_1}{dy}\cdot\frac{d\chi_1}{dz}-\frac{d\psi_0}{dz}\cdot\frac{d\chi_1}{dy}-\frac{d\psi_1}{dz}\cdot\frac{d\chi_0}{dy}-\frac{d\psi_1}{dz}\cdot\frac{d\chi_1}{dy} \\ v=\frac{d\psi_0}{dz}\cdot\frac{d\chi_1}{dx}+\frac{d\psi_1}{dz}\cdot\frac{d\chi_1}{dx}-\frac{d\psi_1}{dx}\cdot\frac{d\chi_0}{dz}-\frac{d\psi_1}{dx}\cdot\frac{d\chi_1}{dz}; \\ w=\frac{d\psi_1}{dx}\cdot\frac{d\chi_0}{dy}+\frac{d\psi_1}{dx}\cdot\frac{d\chi_1}{dy}-\frac{d\psi_0}{dy}\cdot\frac{d\chi_1}{dx}-\frac{d\psi_1}{dy}\cdot\frac{d\chi_1}{dx}; \end{array}\right\} \quad \text{(14)}$$

and all the terms in those expressions, except the 1 in the value of u, represent velocities of disturbance produced in a still mass of liquid by the motion of a solid parallel to x with the velocity -1.

The form of the disturbing solid may be represented by an equation of one or other of the following forms:

$$\psi=0;\quad \chi=0;\quad \mathrm{F}(\psi, \chi)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

In the problems described in the sequel, the first of those expressions is supposed to be used for the figure of the surface of the disturbing solid, viz. $\psi=0$; so that $\psi=b$ with an unlimited series of increasing values of b, expresses the figures of a series of stream-line surfaces lying between successive layers of liquid that enclose the solid within them, like concentric tubes. A series of negative values being given to b, correspond to a set of *internal stream-lines*, which represent currents circulating inside the disturbing solid. In the present investigation, the external stream-lines alone will be considered. The equation $\chi=c$, with a series of values of c, represents a series of stream-line surfaces which meet the surface of the solid ($\psi=0$) edgewise, intersect the surfaces denoted by $\psi=b$, and subdivide the previously mentioned layers of liquid into elementary streams of equal flow.

Two alternative modes of proceeding may be followed in the proposing and solution of problems as to the figures of the stream-line surfaces*. One is as follows: a form is assumed for the function χ, satisfying equations (13) and (12); and thence are deduced, by means of the equations (11), corresponding forms of the function ψ, denoting figures of the disturbing solid and of its enclosing stream-line surfaces; and this is the method which has been followed in previous researches, and which will be followed as regards the quadrifocal stream-lines or cycnogenous neoïds specially treated of in this paper. The other mode of proceeding is to assume for the function ψ a form satisfying equations (13) and (12), and denoting certain figures of the disturbing solid, and of the enclosing stream-line surfaces, and thence to deduce by the aid of the equations (11) the corresponding form and values of the function χ, and the figures of the elementary streams.

From the form of the equations of condition (11) it is easily seen that, if with a given assumed form of either of the functions ψ, χ, there are several forms of the other function which satisfy those equations, then every form obtained by addition or subtraction of those forms will satisfy them also. In symbols, let χ be a given form of one of the functions, and ψ_i any one out of several forms of the other function which, taken along with χ, satisfy the equations; then any function which can be expressed by $\Sigma . \psi_i$ will satisfy them also.

§ 4. *Graphic Construction of Stream-lines.*—Let one side of a piece of paper be taken to represent one of the surfaces whose equation is $\chi=c$. Then the stream-lines which

* Note added in June 1871.—It is to be observed that those methods are tentative only; that is to say, they may fail when tried, and repeated trials may be necessary before a solution is obtained.

are the traces upon that surface of the several surfaces expressed by $\psi=b$* will be represented by lines on that piece of paper; and each of those lines will have an asymptote, being the trace, on the surface $\chi=c$, of a surface whose equation is $\psi_0=b$.

The drawing of such stream-lines is facilitated by the following process invented by Mr. CLERK MAXWELL:—when a function ψ is the sum of two more simple functions, $\psi_0+\psi_1$, draw the series of lines whose equations are $\psi_0=b_0$; then draw the series of lines whose equations are $\psi_1=b_1$; then draw curves diagonally through the angles of the network made by the two former series of curves, in such a manner that at each intersection b_0+b_1 shall be $=b$; the new series of curves will be that represented by the equation $\psi_0+\psi_1=b$. The same process may be extended to curves represented by a function consisting of any number of terms. For example, let the function be one of three terms, $\psi_0+\psi_1+\psi_2$. Draw the two series of lines represented respectively by $\psi_2=b_2$ and $\psi_1=b_1$; through the angles of the network draw the series of lines represented by $\psi_1+\psi_2=b_1+b_2$; then draw a fourth set of lines, being those represented by $\psi_0=b_0$, and through the angles of the network made by the third and fourth series of lines, draw a fifth series of lines, being that represented by

$$\psi_0+\psi_1+\psi_2=b_0+b_1+b_2=b.$$

Figs. 2 and 3 show examples of those processes; and in fig. 4 also the curves have been drawn by means of them, although the network is omitted.

In each case the lines expressed by the function ψ_0 represent a uniform current; and in the figures they are straight and parallel to x. The lines expressed by $\psi-\psi_0$, the sum of the remaining terms of the function, which form a network with the lines of uniform current, may be called *Lines of Disturbance*; for each of them indicates the direction of the motion of disturbance of each particle that it traverses. They are marked with bold dots.

§ 4 A. *Empirical Rule as to the volume enclosed by a Stream-line Surface.*—It has been found by the drawing and measurement of a variety of figures bounded by closed stream-line surfaces, unifocal, bifocal, and quadrifocal, and also by parts of bifocal stream-line surfaces suited for the shapes of vessels, that the following rule gives the volume contained within such a surface to the accuracy of about two per cent.:—multiply the area of midship (or greatest transverse) section by *five sixths* of the longitudinal distance between the pair of transverse sections whose areas are each equal to *one third* of the area of midship section†.

* Note added in June 1871.—The values of b are supposed to be equidifferent.

† This rule was first published as applied to stream-lines in two dimensions, in a treatise entitled 'Ship-building, Theoretical and Practical,' by WATTS, RANKINE, NAPIER, and BARNES: Glasgow, 1866, page 107. Its approximate correctness extends to such extreme cases as a sphere on the one hand and a wave-line bow on the other.

CHAPTER II. *Summary of Principal Properties of previously known Special Classes of Stream-lines.*

§ 5. *Stream-lines in two Dimensions, especially those with two Foci.*—Following the first of the two methods mentioned in § 3, let the simplest of all possible forms be assigned to the function χ, viz. $\chi=z$. This form represents the division of the liquid mass into an indefinite number of layers of uniform thickness, by a series of plane stream-line surfaces parallel to x and to y; and it involves the supposition that all the motions of the particles of liquid take place parallel to the plane of x and y.

The equations (9) in this case become the following:

$$u=\frac{d\psi}{dy};\ v=-\frac{d\psi}{dx};\ w=0. \quad \ldots \quad (16)$$

The equations (11) become the following:

$$\left.\begin{aligned} &\frac{dv}{dz}=-\frac{d^2\psi}{dzdx}=0;\ \frac{du}{dz}=-\frac{d^2\psi}{dydz}=0; \\ &\frac{du}{dy}-\frac{dv}{dx}=\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}=0. \end{aligned}\right\} \quad \ldots \quad (17)$$

The equations (12) and (13) are reduced to the following:

$$\left.\begin{aligned} &\frac{d\psi_0}{dy}=1; \\ &\text{and therefore } \psi_0=y, \text{ and } \psi=y+\psi_1; \end{aligned}\right\} \quad \ldots \quad (18)$$

where ψ_1 is a harmonic function in two dimensions; that is, one fulfilling the condition

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\psi_1=0. \quad \ldots \quad (19)$$

The equations (14) become the following:

$$u=1+\frac{d\psi_1}{dy};\ v=-\frac{d\psi_1}{dx}. \quad \ldots \quad (20)$$

The preceding equations show that the stream-line surfaces are cylindrical (in the general sense), with generating lines parallel to the axis of z, and that they have asymptotic planes parallel to the plane of zx. The traces of those asymptotic planes on the plane xy are a series of equidistant straight lines parallel to the axis of x, and corresponding to an arithmetical series of values of b in the equation $y=b$, being the stream-lines of a uniform current in a plane layer of uniform thickness.

The simplest case of disturbance of such a current by a solid body is that in which the disturbance may be represented by a radiating current, diverging from an axis in the plane of zx, within the solid body and parallel to z, and converging either towards the same axis, or towards a second axis similarly placed; and this is the mode of production of the bifocal stream-lines in two dimensions, or oögenous neoïds, whose properties are investigated in detail in a paper "On Plane Water-lines," published in the Philosophical

Transactions for 1864, page 369. The traces of the axes of divergence and convergence on the plane of xy are called the *foci*. The construction of such bifocal stream-lines is represented by the finer and fainter network of lines in fig. 2. O X and O Y are the axes of coordinates in the plane of projection, which shows a quadrant of each of the stream-lines, the other three quadrants being symmetrical to that shown. The equidistant straight lines parallel to O X are the asymptotes, corresponding to values of $y=b$. A is one of the foci; and the other is situated at an equal distance from O in the contrary direction. The stream-lines of a current in a plane uniform layer diverging from or converging towards a focus are straight, and make equal angles with each other; and their equation is

$$\psi_1=k\tan^{-1}\frac{x-a}{y}=b; \quad \ldots\ldots\ldots\ldots \quad (21)$$

in which $a=$ O A denotes the distance of the focus from the origin, b is a constant having a series of values in arithmetical progression, and k is a constant called the *parameter*; so that $\frac{b}{k}$ is an angle having a series of values in arithmetical progression. This parameter is to be made positive for convergence, and negative for divergence.

If we suppose the diagram extended so as to show both foci, the focus of convergence being in the position $x=+a$, and the focus of divergence in the position $x=-a$, we obtain for the stream-line function representing these motions combined the following expression:

$$\psi_1+\psi_2=k\left(\tan^{-1}\frac{x-a}{y}-\tan^{-1}\frac{x+a}{y}\right)=b. \quad \ldots\ldots \quad (22)$$

The stream-lines or *lines of disturbance* represented by this function are constructed by drawing two similar sets of equiangular radiating straight lines through the two foci, and then drawing curves diagonally through their intersections and through the foci; but as these curves are all circles traversing the foci, it is easier to draw those circles at once, without previously drawing the radiating straight lines; and such is the process described in the paper referred to. The fine arcs which traverse the focus A in fig. 2 are parts of such circular lines of disturbance. Their centres are all in the axis of y; and the radius of any one of them is given by the following formula: let

$$-\frac{b}{k}=\tan^{-1}\frac{x+a}{y}-\tan^{-1}\frac{x-a}{y}=\theta;\ \text{then radius of circle}=a\operatorname{cosec}\theta. \quad . \quad (23)$$

The combination of the divergence and convergence with the uniform current gives, for the stream-lines, the comparatively fine curves in fig. 2, which traverse diagonally the network made by the parallel straight lines and the fine circular lines of disturbance that spread from the focus A. The general equation of those stream-lines is

$$\psi=y+k\left(\tan^{-1}\frac{x-a}{y}-\tan^{-1}\frac{x+a}{y}\right)=y-k\theta=b. \quad \ldots\ldots \quad (24)$$

In the particular case $b=0$, this equation has two roots; viz.

$$\left.\begin{aligned} &y=0, \text{ representing the axis O X, and} \\ &y=k\theta, \text{ representing the oval of which L B in fig. 2 is a quadrant.} \end{aligned}\right\} \quad . \quad . \quad (25)$$

That oval is the trace of the cylindrical surface of a solid which will disturb a uniform current in such a way as to produce the whole series of stream-lines; and it is the only one of those lines which is closed and finite, all the others being infinite and having asymptotes. When the two foci coalesce into one, that oval becomes a circle.

The component comparative velocities are as follows:

$$\left.\begin{aligned} u&=\frac{d\psi}{dy}=1-\frac{k(x-a)}{(x-a)^2+y^2}+\frac{k(x+a)}{(x+a)^2+y^2}; \\ v&=-\frac{d\psi}{dx}=-\frac{ky}{(x-a)^2+y^2}+\frac{ky}{(x+a)^2+y^2}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

In the previous paper already referred to, the parameter here denoted by k is denoted by f; and the comparative velocities here denoted by u and v are denoted by $\frac{u}{c}$ and $\frac{v}{c}$. The origin O is taken midway between the foci for convenience. Should it be placed at unequal distances, let $x=+a'$ for one focus, and $-a''$ for the other; then in the equations, a' is to be put for $-a$, and $+a''$ for $+a$.

Let l denote the half-length O L of the oval stream-line; then by making $u=0$, $y=0$, and $x=l$ in the first of the equations (26), it is found that the following relation exists between the half-length l, the excentricity a, and the parameter k,

$$l^2=a^2-2ka=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26\text{ A})$$

Let y_0 be the greatest half-breadth O B of the oval stream-lines, then we have by equation (24),

$$y_0-2k\tan^{-1}\frac{a}{y_0}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26\text{ B})$$

§ 6. *Stream-line Surfaces of Revolution.*—To obtain by the first method mentioned in § 3 the equations of stream-line surfaces of revolution, the form of the function χ is to be taken so as to represent a series of longitudinal planes cutting each other at equal angles in the axis of x. Hence we have the following expressions:

$$\left.\begin{aligned} &\chi=\tan^{-1}\frac{z}{y}; \ \frac{d\chi}{dx}=0; \\ &\frac{d\chi}{dy}=\frac{-z}{y^2+z^2}; \ \frac{d\chi}{dz}=\frac{y}{y^2+z^2}; \\ &\frac{d^2\chi}{dy^2}=\frac{d^2\chi}{dz^2}=\frac{-2yz}{(y^2+z^2)^2}; \\ &\frac{d^2\chi}{dydz}=\frac{z^2-y^2}{(y^2+z^2)^2}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

As it is sufficient to determine the traces of the stream-line surfaces of revolution in any one of those planes, we may take the plane of xy, for which $z=0$; and then we have the following values:

$$\left.\begin{array}{l} \chi=0;\ \frac{d\chi}{dx}=0;\ \frac{d\chi}{dy}=0; \\ \frac{d\chi}{dz}=\frac{1}{y};\ \frac{d^2\chi}{dy^2}=\frac{d^2\chi}{dz^2}=0; \\ \frac{d^2\chi}{dydz}=-\frac{1}{y^2}. \end{array}\right\} \quad (27\text{A})$$

When the preceding substitutions are made in the equations (9) and (11), they are converted into the following:

The equations (9) become

$$u=\frac{d\psi}{ydy};\ v=-\frac{d\psi}{ydz};\ w=0; \quad (28)$$

and the equations (11) become

$$\left.\begin{array}{l} 0=\frac{dv}{dz}=-\frac{d^2\psi}{ydzdx};\ 0=\frac{du}{dz}=\frac{d^2\psi}{ydydz}\ (\text{and therefore } \frac{d\psi}{dz}=0); \\ 0=\frac{du}{dy}-\frac{dv}{dx}=-\frac{d\psi}{y^2dy}+\frac{1}{y}\left(\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}\right). \end{array}\right\} \quad (29)$$

The same substitutions being made in the equations (12) give the following results:

$$\frac{d\psi_0}{ydy}=1;\ \text{and therefore } \psi_0=\frac{y^2}{2}. \quad (30)$$

This last equation shows that the stream-line surfaces which represent a uniform current, and are asymptotes to the actual disturbed stream-line surfaces, are a series of concentric circular cylinders described about the axis of x, the half squares of whose radii are in arithmetical progression. The traces of such a series of cylindrical surfaces are represented in fig. 3 by the straight lines parallel to the axis O X.

The simplest case of the motion of disturbance produced by a solid of revolution whose axis is the axis of x, is represented by a current diverging symmetrically in all directions from a focus in that axis, and afterwards converging towards another such fcous. The stream-line surfaces of revolution about that axis which represent a diverging or converging current alone, as the case may be, are obviously a series of cones with the focus for their common apex, cutting a spherical surface described about that apex into equal zones. The function which represents the traces on the plane of xy of such a series of conical stream-line surfaces is the following:

$$\psi_1=\pm\frac{k^2}{2}\cdot\frac{x-a}{\sqrt{\{(x-a)^2+y^2\}}}=\frac{k^2\cos\theta}{2}; \quad (31)$$

in which a denotes the distance of the focus from the origin of coordinates and $\pm\frac{k^2}{2}$ is a parameter, to be used with the positive sign for convergence and with the negative sign for divergence.

In the second expression for the function, θ denotes the angle made by the trace of the cone with the axis of x.

To draw a set of those traces, describe a circle about the focus; divide the diameter of that circle which lies along the axis of x into a convenient number of equal parts; through the points of division of the diameter draw ordinates perpendicular to it, cutting the circumference; through the points of division of the circumference draw radii; these will be the required traces of the cones.

For the focus of convergence, let $x=+a$, and for the focus of divergence, let $x=-a$; then the following function represents the lines of disturbance, or stream-lines of the combined motions of divergence and convergence,

$$\psi_1+\psi_2=\frac{k^2}{2}\left\{\frac{x-a}{\sqrt{\{(x-a)^2+y^2)\}}}-\frac{x+a}{\sqrt{\{(x+a)^2+y^2\}}}\right\}=\frac{k^2}{2}(\cos\theta-\cos\theta'); \quad . \quad . \quad (32)$$

in the last of which expressions θ and θ' denote the angles made with the axis of x by the two lines drawn from the point (x, y) to the foci of convergence and divergence respectively. Those lines of disturbance are constructed graphically by drawing two equal and similar sets of radiating straight lines through the foci, as already described, and then drawing curves through the foci, and diagonally through the angles of the network made by the two sets of radiating straight lines. Those curves are already well-known, being the lines of force of a magnet whose poles are at the foci. The fine curves in fig. 3, which spread from the focus A, are examples of them; they were drawn by the method above described, though the radiating straight lines have been omitted from the Plate to prevent confusion.

The stream-lines which are the traces, on the plane of xy, of the stream-line surfaces of revolution, may be constructed, as before, by drawing them diagonally through the angles of the network made by the parallel straight lines in fig. 3 with the lines of disturbance. Their general equation is as follows:

$$\psi=\frac{y^2}{2}+\frac{k^2}{2}\left\{\frac{x-a}{\sqrt{\{(x-a)^2+y^2\}}}-\frac{x+a}{\sqrt{\{(x+a)^2+y^2\}}}\right\}=b, \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

b having a series of values in arithmetical progression. The principal properties of those lines have been stated in the Philosophical Magazine for October 1864; but their detailed investigation has not hitherto been published.

In the particular case $b=0$, equation (33) has two roots, viz.

$$\left.\begin{array}{l} y=0, \text{ representing the axis OX; and} \\ \frac{y^2}{k^2}=\cos\theta'-\cos\theta, \text{ representing the oval of which LB in fig. 3 is a quadrant.} \end{array}\right\} \quad . \quad (34)$$

That oval is the trace of the surface of a solid of revolution which will disturb a uniform current in such a way as to produce the whole series of stream-line surfaces whose traces are expressed by equation (33); and that oval surface of revolution is the only surface of the series which is closed and finite—all the others being indefinitely long, and having asymptotic cylinders expressed by $\frac{y^2}{2}=b$. To avoid confusion these infinite bifocal stream-line surfaces are not shown in fig. 3. They bear a general likeness to those shown in fig. 2.

When the two foci coalesce into one, the disturbing solid becomes a sphere, whose stream-line surfaces were investigated by Dr. HOPPE (Quarterly Journal of Mathematics, March, 1856).

As to the modification of the formulæ required if the origin is not taken midway be tween the foci, see the end of § 5.

The component comparative velocities are as follows:

$$\left.\begin{aligned} u&=\frac{d\psi}{ydy}=1+\frac{k^2}{2}\left\{\frac{-x+a}{\{(x-a)^2+y^2\}^{\frac{3}{2}}}+\frac{x+a}{\{(x+a)^2+y^2\}^{\frac{3}{2}}}\right\}, \\ v&=-\frac{d\psi}{ydx}=\frac{k^2y}{2}\left\{-\frac{1}{\{(x-a)^2+y^2\}^{\frac{3}{2}}}+\frac{1}{\{(x+a)^2+y^2\}^{\frac{3}{2}}}\right\}. \end{aligned}\right\} \quad \text{(35)}$$

Let l denote the half-length O L (fig. 3) of the oval solid. Then by making, in the first of the above equations, $u=0$, $x=l$, and $y=0$, the following relation is found to exist between the half-length, excentricity, and parameter,

$$(l^2-a^2)^2-2k^2la=0. \quad \text{(36)}$$

Let y_0 be the extreme half-breadth OB, then by equation (33) we have

$$y_0^6+a^2y_0^4-4k^4a^2=0. \quad \text{(36 A)}$$

CHAPTER III. *Special Theory of Quadrifocal Stream-lines, or Cycnogenous Neoïds.*

§ 7. *Quadrifocal Stream-lines in general.*—A quadrifocal stream-line is the trace on a longitudinal diametral plane of a quadrifocal-stream-line surface, belonging either to the cylindrical class or to that of surfaces of revolution. The four foci are situated in an axis parallel to the direction of the uniform current which is disturbed by the solid; and, as in the previous chapters, that axis will be taken for the axis of x, and the transverse axis in the plane of projection for the axis of y.

The general equation of a quadrifocal stream-line may be expressed as follows:

$$\psi=\psi_0+\psi_1+\psi_2+\psi_3+\psi_4=b \quad \text{(37)}$$

In that expression ψ_0 is the function representing the uniform current of the velocity 1, which is equal to y or to $\frac{1}{2}y^2$, according as the surfaces are cylindrical or of revolution; ψ_1 expresses the convergence of certain currents towards one of the foci, ψ_2 the divergence of the same currents from a second focus, ψ_3 the convergence of certain currents towards a third focus, ψ_4 the divergence of the same currents from a fourth focus.

The graphic construction of quadrifocal stream-lines is illustrated in figs. 2 and 3. In each of those figures, A is one of the first pair of foci, A′ one of the second pair; the other focus of each pair is supposed to lie at the other side of the origin O, beyond the limits of the drawing.

The lines of disturbance expressed by $\psi_1+\psi_2$, being those due to the first pair of foci, are the fine curves spreading from A. The lines of disturbance expressed by $\psi_3+\psi_4$, being those due to the second pair of foci, are the fine curves spreading from A′. Both those sets of lines were drawn according to the rules given in sections 5 and 6. The *lines of resultant disturbance*, expressed by the function $\psi-\psi_0=\psi_1+\psi_2+\psi_3+\psi_4$, are drawn diagonally through the angles of the network made by the two former sets of lines. They are marked with strong dots in the figures. They all traverse one or other of the foci, A, A′, with the exception of one line, which meets the axis O X at right angles in the point M*.

The actual stream-lines are drawn diagonally through the network made by the lines of uniform current and the lines of disturbance. They are shown by rather strong lines in figs 2 & 3. In each set of quadrifocal stream-lines there is one only that is finite and closed. It corresponds to the value $\psi=b=0$; and it is the trace of the surface of the solid whose disturbing action produces the whole system of stream-lines. It has rounded ends, cutting the axis of x at right angles. In each of the figures 2 & 3, a quadrant of that curve is shown, marked L′ B′. This is the curve which resembles the water-line of Mr. FROUDE's model B, fig. 1, and is therefore properly a *cycnoïd*, or swan-like curve. The equation $\psi=0$ has another root, viz. $y=0$, representing the axis of x. The other stream-lines of the system, lying outside the curve L′ B′, are infinite, and have for asymptotes the stream-lines of the uniform current. They may be called *cycnogenous* stream-lines, as being produced by the cycnoïd stream-line surface.

In a system of bifocal stream-lines there are two independent constants, on which the dimensions and figures of all the lines of the system depend—the excentricity (being half the distance between the foci) and the parameter (as to which see equation 36). In a system of quadrifocal stream-lines, there are five independent constants, viz.:—the two parameters, for the first and second pair of foci respectively; the eccentricity of the first pair of foci; and the distances of the two foci forming the second pair from a point midway between the first pair. If those distances are equal, the cycnoïd curve and each of the stream-lines produced by it have then two ends symmetrical to each other; if unequal, those ends are unsymmetrical. In all the examples shown in the Plate the ends are symmetrical.

In each of the figures 2 and 3, the bifocal oval stream-line marked B L has been described about the first two foci with the same parameter which is assigned to those foci in describing the quadrifocal closed stream-line B′ L′.

Fig. 4 shows a series of cycnoïds, or quadrifocal closed stream-lines, in two dimensions, described about the same four foci. The parameter for the first pair of foci (one of which is marked A) is constant, and is that of the bifocal oval neoïd B L. The parameter for the second pair of foci (one of which is marked A′) was made successively

* In fig. 2 the quadrifocal stream-lines and their lines of disturbance have been engraved on a plate already covered with bifocal stream-lines and their lines of disturbance; and therefore, in order to avoid confusion, some of the quadrifocal lines of resultant disturbance extending from A towards the axis of Y, in the neighbourhood of the point B′, have been omitted. Enough have been drawn to show the principle of their construction. In fig. 3 the series of quadrifocal lines of disturbance is complete.

equal to $\frac{1}{12}$, $\frac{2}{12}$, $\frac{3}{12}$, $\frac{4}{12}$, and $\frac{6}{12}$ of the first parameter; and thus were drawn the five cycnoïd curves marked respectively 1 1, 2 2, 3 3, 4 4, and 6 6. The lines of uniform current and of disturbance used in drawing these curves are omitted in the engraving.

This last figure illustrates the fact that, with a given set of foci, and a given parameter for the inner pair of foci, the cycnoïd becomes leaner and more hollow at the bow as the parameter for the outer pair of foci diminishes; also that, with large values of the second parameter, that curve is convex throughout, like the line marked 66; and that for some intermediate value the hollowness just vanishes, as is very nearly the case in the line marked 4 4. It is obvious that any degree of fineness may be given to the entrance by increasing the distance of the second foci from the first, and at the same time using a small second parameter.

§ 8. *Cylindric Cycnoïds.—Forms and Velocities of Streams.*—The equation of a system of quadrifocal stream-lines in two dimensions is as follows,

$$\psi=y+k\left(\tan^{-1}\frac{x-a}{y}-\tan^{-1}\frac{x+a}{y}\right)+k'\left(\tan^{-1}\frac{x-a'}{y}-\tan^{-1}\frac{x+a''}{y}\right)=b, \quad . \quad . \quad (38)$$

in which k and k' are the parameters for the inner and outer pairs of foci respectively, a is the excentricity of the inner pair of foci, and a' and a'' are the distances of the outer pair of foci from the origin in opposite directions. The equation of the cycnoïd curve, or trace of the surface of the cylindric solid which generates the series of stream-lines, is $\psi=b=0$. If that solid is symmetrical-ended, we have $a''=a'$. The components of the comparative velocity of a stream at a given point (x, y) are given by the following equations, in which, for brevity's sake, the following notation is used:

$$(x-a)^2+y^2=r_1^2;\ (x+a)^2+y^2=r_2^2:$$
$$(x-a')^2+y^2=r_3^2;\ (x+a'')^2+y^2=r_4^2.$$
$$\left.\begin{aligned} u&=\frac{d\psi}{dy}=1-k\left(\frac{x-a}{r_1^2}-\frac{x+a}{r_2^2}\right)-k'\left(\frac{x-a'}{r_3^2}-\frac{x+a''}{r_4^2}\right); \\ v&=-\frac{d\psi}{dx}=-ky\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)-k'y\left(\frac{1}{r_3^2}-\frac{1}{r_4^2}\right). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (39)$$

At the extreme breadth of the space bounded by a given stream-line we have $v=0$; and when the cycnoïd is symmetrical-ended, the longitudinal component u at the same point takes the following value, found by making $x=0$,

$$u_0=1+\frac{2ka}{a^2+y_0^2}+\frac{2k'a'}{a'^2+y_0^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39\text{ A})$$

where y_0 denotes the greatest ordinate or "midship half-breadth" of the stream-line under consideration.

§ 9. *Cylindric Cycnoïds.—Extreme Dimensions.*—The extreme length of a cylindric cycnoïd is made up of the distances of its two rounded ends, where it cuts the axis of x, from the origin of coordinates. Let l be one of those distances; in the expression for

u, equation (39), make $x=l$, $y=0$, $u=0$; then we have the following equation,

$$0=1-k\left(\frac{1}{l-a}-\frac{1}{l+a}\right)-k'\left(\frac{1}{l-a'}-\frac{1}{l+a''}\right),$$

which by ordinary reductions gives the following biquadratic equation:

$$\left.\begin{array}{l} 0=l^4-l^3(a'-a'')-l^2(a^2+a'a''+2ka+k'(a'+a'')) \\ +l(a^2-2ka)(a'-a'')+a^2a'a''+2kaa'a'' \\ +k'a^2(a'+a''). \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Of the four roots of this equation, the two greatest, positive and negative respectively, belong to the cylindric cycnoïd; and the sum of their arithmetical values is its length. The two least, positive and negative, belong to an internal stream-line, which is also a closed curve. It passes outside and near to the inner foci, and inside the outer foci, and it is foreign to the purpose of the present investigation.

When the two outer foci are equidistant from the inner foci (that is, when $a''=a'$), equation (40) becomes a quadratic equation in l^2; that is to say, we have

$$0=l^4-l^2(a^2+a'^2+2ka+2k'a')+a^2a'^2+2kaa'^2+2k'a'a^2. \quad . \quad . \quad . \quad . \quad (40\text{ A})$$

For brevity's sake, let

$$a^2+2ka=\lambda^2, \quad a'^2+2k'a'=\lambda'^2,$$

being in fact, according to equation (26 A), the values of l^2 for two bifocal oval neoïds, with the respective excentricities a and a', and parameters k and k'. Then the solution of equation (40 A) is as follows:

$$l^2=\frac{\lambda^2+\lambda'^2}{2}\pm\sqrt{\left\{\frac{(\lambda^2-\lambda'^2)^2}{4}+4kk'aa'.\right\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40\text{ B})$$

The greater root is the square of the half-length of the cycnoïd; the lesser root belongs to the internal stream-line already mentioned.

The method of finding the extreme half-breadth in a cycnoïd with unsymmetrical ends, is to make $\psi=0$ in equation (38), and $\frac{v}{y}=0$ in the second equation (39), and, from the pair of equations so obtained, to deduce x and y by elimination. When the ends of the cycnoïd are symmetrical, the extreme half-breadth is midway between the foci; hence, making $x=0$ in equation (38), we have the following transcendental equation,

$$0=y_0-2k\tan^{-1}\frac{a}{y_0}-2k'\tan^{-1}\frac{a'}{y_0}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

from which y_0 is to be calculated by approximation.

§ 10. *Cycnoïds of Revolution.—Forms and Velocities of Streams.*—The equation of a series of cycnogenous or quadrifocal stream-lines of revolution is as follows:

$$\psi=\frac{y^2}{2}+\frac{k^2}{2}\left(\frac{x-a}{r_1}-\frac{x+a}{r_2}\right)+\frac{k'^2}{2}\left(\frac{x-a'}{r_3}-\frac{x+a''}{r_4}\right)=b; \quad . \quad . \quad . \quad . \quad . \quad (42)$$

in which r_1, r_2, r_3, and r_4 have the same meaning as in equation (38); that is, they are the distances of the point (x, y) from the four foci respectively. The equation of the cycnoïd of revolution which produces the series of stream-lines is $\psi=b=0$; and this equation has two roots, viz. $y=0$, denoting the axis of x, and

$$y^2=k^2(\cos\theta_2-\cos\theta_1)+k'^2(\cos\theta_4-\cos\theta_3), \quad \text{(42 A)}$$

in which θ_1, θ_2, θ_3, and θ_4 denote the angles made with the axis of x by lines drawn from the point (x, y) to the four foci.

The component comparative velocities are as follows:

$$\left.\begin{aligned} u&=\frac{d\psi}{ydy}=1+\frac{k^2}{2}\left(\frac{-x+a}{r_1^3}+\frac{x+a}{r_2^3}\right)+\frac{k'^2}{2}\left(\frac{-x+a'}{r_3^3}+\frac{x+a''}{r_4^3}\right);\\ v&=-\frac{d\psi}{ydx}=\frac{k^2y}{2}\left(-\frac{1}{r_1^3}+\frac{1}{r_2^3}\right)+\frac{k'^2y}{2}\left(-\frac{1}{r_3^3}+\frac{1}{r_4^3}\right).\end{aligned}\right\} \quad \text{(43)}$$

When the two ends of the solid are symmetrical, we have $a''=a'$; and the value of u at the midship section, where $v=0$ and $x=0$, is as follows,

$$u_0=1+\frac{k^2a}{(a^2+y_0)^{\frac{3}{2}}}+\frac{k'^2a'}{(a'^2+y_0)^{\frac{3}{2}}}; \quad \text{(43 A)}$$

in which y_0 is the midship half-breadth.

§ 11. *Cycnoïds of Revolution.—Extreme Dimensions.*—Let l denote the distance from the origin of one of the points where the cycnoïd surface of revolution cuts the axis of x. Then, in the first of the equations (43), making $u=0$, $y=0$, $x=l$, we obtain the following equation of the eighth order,

$$\left.\begin{aligned} 0=&(l^2-a^2)^2\,.\,(l-a')^2(l+a'')^2-2k^2la(l-a')^2(l+a'')^2\\ &-\frac{k'^2}{2}\left\{2l(a''+a')+(a''^2-a'^2)\right\}\cdot(l^2-a^2)^2.\end{aligned}\right\} \quad \text{(44)}$$

The greatest positive and greatest negative real roots of this equation give the ends of the cycnoïd; the other real roots belong to internal stream-lines.

When the ends of the solid are symmetrical, so that $a''=a$, the preceding equation becomes

$$0=(l^2-a^2)^2(l^2-a'^2)^2-2k^2la(l^2-a'^2)^2-2k'^2la'(l^2-a^2)^2. \quad \text{(44 A)}$$

The greatest half-breadth and its position are to be found in the general case, as before, by deducing values of y and x by elimination from the pair of equations $\psi=0$, $v=0$. When the ends of the solid are symmetrical, the greatest half-breadth is at the origin; hence, making $x=0$, $v=0$, we have the following equation,

$$0=y_0^2-\frac{2k^2a}{\sqrt{(a^2+y_0^2)}}-\frac{2k'^2a'}{\sqrt{(a'^2+y_0^2)}}; \quad \text{(45)}$$

which, when reduced to the form of an algebraic equation with y_0^2 for the unknown quantity, is of the eighth order, as follows:

$$\left.\begin{aligned} 0=&y_0^8(y_0^2+a^2)^2(y_0^2+a'^2)^2+16k^8a^4(y_0^2+a'^2)^2\\ &+16k'^8a'^4(y_0^2+a^2)^2-8k^4a^2y_0^4(y_0^2+a^2)(y_0^2+a'^2)^2\\ &-8k'^4a'^2y_0^4(y_0^2+a^2)^2(y_0^2+a'^2)\\ &+16k^4k'^4a^2a'^2(y_0+a^2)(y_0^2+a'^2).\end{aligned}\right\} \quad \text{(45 A)}$$

Equation (45) may be used to solve the following problem:—Given the midship half-breadth y_0, the excentricities of the two pairs of foci a, a', and the inner parameter k^2; to find the outer parameter k'^2.

CHAPTER IV. *Dynamical Propositions as to Stream-line Surfaces.*

§ 12. *Resultant Momentum.*—The resultant momentum, parallel to x, of any part of a given elementary stream is equal to that of an undisturbed part of the same stream whose length, projected on the axis of x, is the same. For let σ_0 be the sectional area of an undisturbed part of such a stream and 1 its velocity; then $\sigma_0 dx$ is the momentum of an elementary part of its length.

Let dx also be the projection on the axis of x of an elementary part of the same stream, when disturbed, σ the sectional area of that part on a plane normal to x, and u its component velocity parallel to x; then its component momentum parallel to x is $u\sigma dx$. But $u\sigma$ is the volume of flow along the elementary stream, which is uniform and $=\sigma_0$; therefore

$$u\sigma dx=\sigma_0 dx;$$

so that the component momentum parallel to x of any part of an elementary stream is simply

$$\sigma_0(x_2-x_1);$$

in which x_2 and x_1 are the values of x for its two ends. Consider now an elementary stream of indefinitely great length, so that its two ends lie in one straight line parallel to x, and are at so great a distance from the disturbing solid that its action on the particles at those ends vanishes. The resultant momentum of that stream is the same as if it were undisturbed; and such being the case for every elementary stream, is the case for the whole mass of liquid. This conclusion is expressed by the following equations, in which the integrations extend throughout the whole liquid mass outside the surface of the disturbing solid

$$\left.\begin{array}{l}\iiint(u-1)dx\,dy\,dz=0;\\ \iiint v dx\,dy\,dz=0;\ \iiint w dx\,dy\,dz=0.\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

The resultant momentum $\iiint u dx\,dy\,dz$ is that of the liquid relatively to the solid, considered as fixed.

If we next consider the centre of mass of the liquid as fixed, the resultant momentum of the liquid becomes

$$\iiint(u-1)dx\,dy\,dz=0;$$

and that of the solid relatively to the liquid, per unit of velocity and density, is represented by $-$D, D denoting the *displacement* of the solid (that is, the volume of liquid which it displaces, and also the mass of the solid supposed equal to that of the displaced liquid).

Thirdly, let the *common centre of mass* of the liquid and solid be taken as a fixed point, and let the momenta of the liquid and solid relatively to that point be taken. Those momenta are equal and opposite—that of the liquid being positive, and that of the solid negative. The velocity of the centre of mass of the liquid relatively to the solid being still taken as unity, its velocity relatively to the common centre is expressed as follows, L being the total mass of the liquid,

$$\frac{D}{L+D}. \quad \text{(46 A)}$$

The velocity of the solid relatively to the common centre is

$$\frac{L}{L+D}; \quad \text{(46 B)}$$

and the respective equal and opposite momenta of the solid and liquid relatively to the same point are expressed by

$$\pm\frac{LD}{L+D}. \quad \text{(46 C)}$$

When the mass of liquid L becomes indefinitely great, $\frac{D}{L+D}$ becomes indefinitely small, $-\frac{L}{L+D}$ approximates indefinitely to -1, and $\pm\frac{LD}{L+D}$ to $\pm D$; but notwithstanding these indefinitely close approximations, it is necessary to bear in mind that (as is implied in equation 46) the component longitudinal velocity of current u is taken *relatively to the centre of mass of the liquid*, and not relatively to the common centre of mass, the corresponding component relatively to the common centre being

$$u+\frac{D}{L+D}.$$

If the liquid is absolutely free from stiffness and friction, the *resultant pressure* exerted between it and the solid in a horizontal direction is obviously equal to nothing, so long as the velocity is uniform, and only acquires a value in the event of acceleration or retardation; which value is expressed by the rate of change per second in the equal and opposite momenta $\pm\frac{LD}{L+D}$.

To adapt the formulæ of this and the ensuing sections to other velocities and densities than those denoted by unity, let $-V$ be the velocity of the solid, and ϱ the density of the liquid; then quantities denoting velocities are to be multiplied by V, those denoting masses by ϱ, those denoting momentum by $V\varrho$, those denoting heights due to velocities by V^2, those denoting energy, and those denoting intensity of pressure, by $V^2\varrho$.

It is to be observed that, according to the notation of this paper, motion ahead is treated as negative, and motion astern as positive, the latter being the direction of the motion of the liquid relatively to the solid.

§ 13. *Energy of Currents and of Disturbance.*—The energy of the motion of the liquid mass contained within a given space may be taken either relatively to the disturbing solid, considered as fixed, in which case it may be called the *energy of current*, or relatively to the undisturbed liquid, in which case it may be called the *energy of disturbance.* Assuming unity, as before, for the values of the undisturbed velocity and of the density, it is obvious that the energy of current in an elementary space of the volume $dx\,dy\,dz$ is

$$\tfrac{1}{2}(u^2+v^2+w^2)dx\,dy\,dz, \quad \ldots \ldots \quad \text{(A)}$$

and that the energy of disturbance is

$$\tfrac{1}{2}(u^2+v^2+w^2-2u+1)\,dx\,dy\,dz. \quad \ldots \ldots \quad \text{(B)}$$

To find the total energy of current, or of disturbance, as the case may be, in a given finite space, the one or the other of the two preceding expressions is to be integrated throughout that space. In order to solve questions of this kind, recourse must be had to the *velocity-function* (ϕ) well known in hydrodynamics, and already referred to in § 2, equations (1) to (7), and in § 3, equation (9), as representing by its values a series of surfaces which cut all the elementary streams at right angles—and especially to a property of that kind of function which was first demonstrated by GREEN, in his Essay on Potential Functions, and which is expressed as follows:—Let ϕ be a function of x, y and z, which fulfils the condition

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=0\,;$$

let $d\sigma$ be an elementary part of the bounding surface of an enclosed space, and let $\frac{d}{dn}$ denote differentiation relatively to the normal to that elementary part, dn being positive outwards; then (under certain limitations which do not affect the subject of the present paper)* we have

$$\iiint\left(\frac{d\phi^2}{dx^2}+\frac{d\phi^2}{dy^2}+\frac{d\phi^2}{dz^2}\right)dx\,dy\,dz=\iint\phi\,\frac{d\phi}{dn}\,d\sigma, \quad \ldots \ldots \quad \text{(C)}$$

the double integral extending to all parts of the bounding surface. Observing now that

$$u=\frac{d\phi}{dx},\ v=\frac{d\phi}{dy},\ w=\frac{d\phi}{dz},$$

let E_C denote the energy of current, and E_D the energy of disturbance, within a given space, corresponding to the undisturbed velocity 1 and density 1; then we have

$$E_C=\tfrac{1}{2}\iint\phi\,\frac{d\phi}{dn}\,d\sigma\,; \quad \ldots \ldots \quad (47)$$

$$E_D=\tfrac{1}{2}\iint\phi\,\frac{d\phi}{dn}\,d\sigma-\iiint\left(\frac{d\phi}{dx}-\tfrac{1}{2}\right)dx\,dy\,dz. \quad \ldots \ldots \quad (47\text{A})$$

It is next to be observed that, because the velocity-function ϕ expresses a series of surfaces cutting all the stream-lines at right angles, the coefficient $\frac{d\phi}{dn}$ (denoting the component velocity normal to the elementary surface $d\sigma$) is *nothing* for all bounding surfaces and parts of bounding surfaces that coincide with stream-line surfaces,—and therefore that, in finding the integral E_C which expresses the energy of current within a given

* As to the limitations to which this proposition is subject, see a paper by HELMHOLTZ, in CRELLE's Journal for 1858, "Ueber Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen;" also THOMSON en Vortex-Motion, Trans. Roy. Soc. Edin. 1867–68, pp. 239 *et seqq.*

space, it is necessary to take into account *those boundaries only of that space which intersect the stream-lines.*

The values of the velocity-function ϕ for quadrifocal stream-line surfaces are obviously the following:—For cylindrical stream-line surfaces,

$$\phi=x+k \text{ hyp} \log \frac{r_2}{r_1}+k' \text{ hyp} \log \frac{r_4}{r_3}; \quad \ldots \quad (48)$$

for stream-line surfaces of revolution,

$$\phi=x+\frac{k^2}{2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+\frac{k'^2}{2}\left(\frac{1}{r_3}-\frac{1}{r_4}\right); \quad \ldots \quad (49)$$

in which r_1, r_2, r_3, and r_4 denote, as before, the distances of a point from the four foci. These expressions may be made applicable to bifocal surfaces by making $k'=0$, and might be extended to surfaces with any number of pairs of foci by increasing the number of terms and parameters.

When a pair of foci coalesce, the function of r belonging to those foci is to undergo the operation $-\text{A}\frac{d}{dx}$, in which A is an arbitrary constant of one dimension—thus giving, for cylindrical surfaces, a term of the form $-\frac{k\text{A}x}{r^2}$, and for surfaces of revolution a term of the form $-\frac{k^2\text{A}x}{2r^3}$.

In the foregoing investigations, and in their appplications which are to follow, the energy of disturbance is taken relatively to the centre of mass of the liquid. If taken relatively to the common centre of mass of the liquid and solid, it would be increased by a quantity whose value for the whole mass of liquid, per unit of undisturbed velocity and of density, is

$$\frac{\text{D}^2\text{L}}{2(\text{L}+\text{D})^2}=\frac{\text{D}^2}{2(\text{L}+\text{D})^2}\iiint dx\,dy\,dz; \quad \ldots \quad (49\text{ B})$$

but when the extent of the liquid is unlimited, that quantity vanishes as compared with the quantity given by equation (47 A).

§ 14. *Energy in an Elementary Stream.*—In order to apply the principles of the preceding article to the whole or to a given part of an elementary stream, let σ_0 be the transverse sectional area of that stream when undisturbed, measured on a plane normal to x, σ the sectional area on such a plane at a given point, x_1 and x_2 the values of x, σ_1 and σ_2 the values of σ, and ϕ_1 and ϕ_2 the values of ϕ, for the two ends of the part of the stream under consideration; and let x_2 be greater than x_1. Then the energy of current, per unit of undisturbed velocity and of density, is found by taking the integral in equation (47) for those two ends only; that is to say,

$$\text{E}_\text{C}=\tfrac{1}{2}\phi_2\frac{d\phi_2}{dx_2}\sigma_2-\tfrac{1}{2}\phi_1\frac{d\phi_1}{dx_1}\sigma_1;$$

but $\frac{d\phi}{dx}\sigma=u\sigma=\sigma_0$; and therefore we have simply, for the energy of current,

$$\text{E}_\text{C}=\frac{\sigma_0}{2}(\phi_2-\phi_1). \quad \ldots \quad (50)$$

The energy of disturbance per unit of velocity and density is expressed by

$$\mathrm{E_D}=\mathrm{E_C}-\int_{x_1}^{x_2}(u-\tfrac{1}{2})\sigma dx)=\frac{\sigma_0}{2}\left\{\phi_2-\phi_1-2x_2+2x_1+\int_{x_1}^{x_2}\frac{dx}{u}\right\}. \quad . \quad . \quad . \quad . \quad (51)$$

§ 15. *Total Energy of Disturbance.*—In order to find the total energy of disturbance throughout the indefinitely extended mass of liquid, the most convenient method is to find the limit to which $\mathrm{E_D}$ in equation (47 A) approximates when the integrals are taken throughout a circular cylinder for cylindric cycnoïds, or a sphere for cycnoïds of revolution, and the radius of the cylinder or of the sphere is indefinitely increased. The coefficient $\frac{d\phi}{dn}$ is $=0$ at every point of the surface of the disturbing solid; therefore no integration has to be performed over that surface. The triple integral in the second term of the equation, viz.

$$-\iiint(u-\tfrac{1}{2})dx\,dy\,dz,$$

may be simplified by considering that, because the integration extends throughout the unlimited mass of the liquid, we have, by equation (46),

$$\iiint(u-1)dx\,dy\,dz=0,$$

and consequently

$$-\iiint(u-\tfrac{1}{2})dx\,dy\,dz=-\tfrac{1}{2}\iiint dx\,dy\,dz.$$

Now this is obviously the half difference, with the sign reversed, between the volume of the indefinitely large cylinder or sphere, as the case may be, and the displacement or volume of the disturbing solid, denoted by D. Moreover, in the first term of the equation, we have $dn=dr$; $\int d\sigma=rd\theta$ for a cylinder, or $2\pi r^2\sin\theta d\theta$ for a sphere, θ being the angle which r makes with the axis of x; and the limits of integration are from $\theta=0$ to $\theta=2\pi$ for a cylinder, and from $\theta=0$ to $\theta=\pi$ for a sphere. Hence we have the following expressions:—For indefinitely deep cylindrical solids,

$$\mathrm{E_D}=\tfrac{1}{2}\int_0^{2\pi}\left(\phi\frac{d\phi}{dr}-\frac{r}{2}\right)rd\theta+\frac{\mathrm{D}}{2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

for solids of revolution,

$$\mathrm{E_D}=\tfrac{1}{2}\int_0^{\pi}\left(\phi\frac{d\phi}{dr}-\frac{r}{3}\right)2\pi r^2\sin\theta d\theta+\frac{\mathrm{D}}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

In taking the values of ϕ and $\frac{d\phi}{dr}$ corresponding to an indefinitely great value of r, it is to be observed that the distance $2a$, or $a'+a''$, between a given pair of foci, becomes indefinitely small compared with r, and that consequently, if F be a function of the distance from a focus, and $\Delta\mathrm{F}$ the difference of its values for a pair of foci whose distance apart is $2a$, we are to make

$$\Delta\mathrm{F}\text{ sensibly}=-2a\frac{d\mathrm{F}}{dx}.$$

Hence $\left(\text{observing that } \frac{dx}{dr}=\frac{x}{r}\right)$ we have for an indefinitely large cylinder,

$$\left.\begin{aligned}\phi&=x-\Sigma\,.\,\frac{2kax}{r^2},\\ \frac{d\phi}{dr}&=\frac{x}{r}+\Sigma\,.\,\frac{2kax}{r^3};\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(54)$$

and for an indefinitely large sphere,

$$\left.\begin{aligned}\phi&=x-\Sigma\,.\,\frac{k^2ax}{r^3},\\ \frac{d\phi}{dr}&=\frac{x}{r}+\Sigma\,.\,\frac{2k^2ax}{r^4};\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(55)$$

in each of which expressions Σ denotes the summation of terms belonging to the several pairs of foci, if there are more than one pair—each term containing its proper parameter, k or k^2, and its proper double excentricity, $2a(=a'+a''$ when those two distances are unequal).

Substituting $\cos\theta$ for $\frac{x}{r}$, the functions within brackets in the integrals of equations (52) and (53) are found to have the following values:—

Cylinder:

$$\phi\frac{d\phi}{dr}-\frac{r}{2}=r(\cos^2\theta-\tfrac{1}{2}-\text{terms in }\frac{1}{r^4}\ \&\text{c.}).\ \ldots\ldots\quad(56)$$

Sphere:

$$\phi\frac{d\phi}{dr}-\frac{r}{3}=r(\cos^2\theta-\tfrac{1}{3}+\Sigma\,.\,\frac{k^2a\cos^2\theta}{r^2}-\text{terms in }\frac{1}{r^6}\ \&\text{c.}).\ \ldots\quad(57)$$

The terms in $\frac{1}{r^4}$ and higher powers of $\frac{1}{r}$ vanish, because of the indefinite increase of r. The terms in $\cos^2\theta-\frac{1}{2}$ and $\cos^2\theta-\frac{1}{3}$ disappear from the integration. Hence the integral in equation (52) vanishes altogether; and that in equation (53) has for its value

$$\tfrac{1}{2}\int_0^\pi\Sigma\left(\frac{k^2a\cos^2\theta}{r^2}\right).\,2\pi r^2\sin\theta d\theta=-\frac{2\pi}{3}\,\Sigma k^2a;\ \ldots\ldots\quad(58)$$

so that we obtain finally, for *the total energy of disturbance per unit of velocity and of density*, if the disturbing solid is an indefinitely deep cylinder,

$$E_D=\tfrac{1}{2}D;\ \ldots\ldots\ldots\quad(59)$$

and if it is a solid of revolution,

$$E_D=\tfrac{1}{2}\left(D-\frac{4\pi}{3}\,\Sigma k^2\,a\right).\ \ldots\ldots\ldots\quad(60)$$

The *ratio borne by the total energy of disturbance to the energy of the disturbing solid* is:—

for indefinitely deep cylinders,

$$\frac{2E_D}{D}=1;\ \ldots\ldots\ldots\quad(59\text{ A})$$

for solids of revolution,

$$\frac{2E_D}{D}=1-\frac{4\pi}{3D}\cdot\Sigma k^2a; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60\ \text{A})$$

observing, in the last expression, that for any pair of foci whose distances from the origin a' and a'' are unequal, the mean of those distances, $\frac{a'+a''}{2}$, is to be taken as the value of a.

When the disturbing solid is a sphere of the radius l, its displacement is $D=\frac{4\pi l^3}{3}$. It has one focus at its centre, produced by the coalescence of a pair of foci; k^2 becomes indefinitely great, and a indefinitely small; but their product has a finite value, $k^2a=\frac{l^3}{2}$. Hence in this case we have

$$\frac{2E_D}{D}=1-\tfrac{1}{2}=\tfrac{1}{2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60\ \text{B})$$

that is to say, *the total energy of the disturbance produced by a sphere is equal to half the energy of the sphere.*

When the solid is an oval or bifocal neoïd of revolution, and the excentricity a increases indefinitely as compared with the parameter k^2, the displacement approximates upwards towards that of a cylinder of revolution of the length $2a$ and transverse section $2\pi k^2$ (that is, towards $4\pi k^2a$); so that in this case we have for the upper limit of the ratio of the total energy of disturbance to the energy of the solid, the following value:—

$$\frac{2E_D}{D}=1-\tfrac{1}{3}=\tfrac{2}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60\ \text{C})$$

For all neoïds of revolution, oval and cycnoïd, the ratio in question lies between the limits $\frac{1}{2}$ and $\frac{2}{3}$. Its value in any particular case may always be determined to any required degree of approximation by constructing the figure of the disturbing solid and measuring its displacement. For example, in fig. 3 it is found to be, for the oval neoïd of revolution L B, 0·56; and for the cycnoïd of revolution L′ B′, 0·6 nearly.

The principles of this and the three preceding sections (§ 12, 13, and 14) are applicable not only to bifocal, quadrifocal, and other stream-line surfaces having foci situated in one axis, but to all stream-line surfaces which can be generated by combining a uniform current with disturbances generated by pairs of foci arranged in any manner whatsoever, or having, instead of detached focal points, *focal spaces*; the disturbance-functions belonging to which are to be found by integrating the corresponding functions belonging to the points contained in those spaces, a process similar to that of finding the potential of a solid*.

§ 16. *Disturbance of Pressure and Level.*—It is well known that in all cases of the steady flow of a liquid, the sum of the height due to velocity, and the height due to elevation and pressure combined, is constant in a given elementary stream; that is to

* Note by the Reporter.—See paper, Professor C. NEUMANN, in CRELLE'S Journal for 1861, on the equation $\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=0$.

say, let h_0 be the *head*, or height due to elevation and pressure, in a given elementary stream, at a point where the velocity is that of the undisturbed uniform current; let V, as before, denote that velocity, so that uV, vV, and wV are the components of the velocity at any other point; then, at that other point, the head is given by the following equation,

$$h=h_0+\frac{V^2}{2g}(1-u^2-v^2-w^2); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

and the following difference may be called the *disturbance of head*,

$$h-h_0=\frac{V^2}{2g}(1-u^2-v^2-w^2).$$

Some of the general consequences of this principle have been pointed out in the paper "On Plane Water-lines" already referred to; and its bearing on the laws of the resistance of ships has been shown in a paper "On the Computation of the probable Engine-power and Speed of proposed Ships," published in the Transactions of the Institution of Naval Architects for 1864.

In connexion with the subject of the present paper, it is sufficient to state that, when a current of a perfect liquid of unlimited extent in all directions flows past a solid, the disturbance of head takes the form of variation of pressure only, the energy of a given particle of an elementary stream changing its form between energy of motion and energy of pressure as the velocity varies—so that points of minimum velocity of current are points of maximum pressure, and points of maximum velocity of current are points of minimum pressure,—but that where the current is bounded above by a free upper surface, exposed to the air, that surface continues to be everywhere a surface of uniform pressure, and the disturbances of head take the form of disturbances of level, places of minimum velocity being marked by a swell, and those of maximum velocity by a hollow. For example, when a floating solid body, as a ship, moves through still water, the surface of the water is raised at those points where the particles of water are pushed or drawn ahead by the ship, and depressed at those points where they run astern past her sides in order to fill up the space in her wake.

The aggregate disturbance of head throughout the whole liquid mass is expressed as follows,

$$\iiint(h-h_0)dx\,dy\,dz=-\frac{V^2}{2g}\cdot E_D, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

being obviously equal, but of contrary sign, to the total energy of disturbance per unit of density (see equation 47 A).

Let the whole volume of the liquid mass be denoted by $L=\iiint dx\,dy\,dz$; then

$$\frac{\iiint(h-h_0)dx\,dy\,dz}{\iiint dx\,dy\,dz}=-\frac{V^2E_D}{2gL} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

expresses a depression of the centre of gravity of that mass relatively to the surface of the liquid at an indefinite distance from the disturbing solid—in other words, *an eleva-*

tion of the surface of the liquid at an indefinite distance from the solid above the centre of mass of the liquid; so that the disturbance of head at any point *relatively to that centre of mass* is expressed as follows:—

$$h-h_0+\frac{V^2E_D}{2gL}. \qquad (64)$$

The last term vanishes when the volume of liquid L increases indefinitely.

When the trace of a disturbing solid, together with its external stream-lines and lines of disturbance, has been drawn, as in figs. 2 and 3, the manner in which the disturbances of motion and of head vary at different points may be represented to the eye by means of a diagram like fig. 5, constructed as follows. Draw a straight line A B to represent the velocity of the undisturbed current (equal and opposite to the velocity of the ship). From A draw a series of straight lines, such as A C, A C′, A C″, parallel to a series of tangents at a series of points in the trace of the solid. From B draw a series of straight lines, such as B C, B C′, B C″, parallel to the tangents of the lines of disturbance at the same series of points, cutting the first-mentioned series of lines in C, C′, C″, Then in each of the triangles in the diagram, such as A B C, corresponding to a given point in the trace of the solid, B C will represent the direction and velocity of the disturbance, A C the direction and velocity of the elementary stream of liquid relatively to the solid; and the disturbance of head, positive upwards, will be expressed by $\frac{AB^2-AC^2}{2g}$.

At the points marked L and L′ in figs. 2, 3, and 4, the disturbance of head is simply the height due to the velocity of the disturbing solid.

When the disturbances of head, as in a liquid with a free upper surface, take the form of disturbances of level, they produce two effects—alteration of the forms and motion of the elementary streams, and the formation of waves; which waves may give rise to a particular kind of resistance. In the present paper it is assumed that the dimensions of the disturbing solid are so large, or its motion so slow, that the effects of the disturbances of level on the forms and motions of the elementary streams may be neglected; and the investigation in the ensuing sections is confined to the action of those disturbances in producing waves and wave-resistance.

§ 17. *Virtual Depth and Speed of Waves.*—The term *virtual depth of longitudinal disturbance*, or, more briefly, *virtual depth*, is used to denote the depth found by integrating the velocity of longitudinal disturbance throughout a vertical column of a liquid mass, and dividing the integral by the value of that velocity at the free upper surface of the mass. For example, let $u-1$ be the velocity of longitudinal disturbance in a given indefinitely slender vertical column at the depth z, and u_1-1 its value at the surface; and let Z be the virtual depth; then

$$Z=\frac{\int(u-1)dz}{u_1-1}; \qquad (65)$$

when the column is a prism of finite dimensions, the *mean virtual depth* is as follows:—

$$Z_m = \frac{\iiint(u-1)dx\,dy\,dz}{\iint(u_1-1)dx\,dy}. \quad \ldots \quad (66)$$

When the disturbance is that produced by the longitudinal advance of a solid whose figure is a stream-line surface of revolution, with any number of pairs of foci, floating immersed to the axis in a liquid of indefinite depth, the integrations indicated in the preceding equations give the following results:—

Virtual depth at a given point,

$$Z = \frac{\Sigma k^2\left(\frac{a-x}{r_1^2}+\frac{a+x}{r_2^2}\right)}{\Sigma k^2\left(\frac{a-x}{r_1^3}+\frac{a+x}{r_2^3}\right)}, \quad \ldots \quad (67)$$

the notation being the same as in equations (48) and (49).

Mean virtual depth throughout the whole mass,

$$Z_m = \frac{D}{S}; \quad \ldots \quad (68)$$

in which D is the displacement of the floating solid, and S the area of its *water-section* (that is, of its horizontal section in the plane of the surface of the water); so that the mean virtual depth is equal simply to the mean depth of immersion of the solid. Here it must be explained that, when the disturbances relatively to the centre of mass of the liquid are integrated, equation (68) takes the form $Z_m = \frac{0}{0}$, and that the value $\frac{D}{S}$ is obtained by taking the disturbances relatively to the common centre of mass of the liquid and solid.

At the two ends of the floating solid, where $x=l$ and $y=0$, the virtual depth takes the following value,

$$Z_l = \frac{\Sigma \cdot \frac{k^2 a}{l^2-a^2}}{2l\Sigma \cdot \frac{k^2 a}{(l^2-a^2)^2}}. \quad \ldots \quad (69)$$

When there is but one pair of foci, this is reduced to $\frac{l^2-a^2}{2l}$.

At the midship section (B and B′ in the figures) the virtual depth is

$$Z_{y_0} = \frac{\Sigma \cdot \frac{k^2 a}{a^2+y_0^2}}{\Sigma \cdot \frac{k^2 a}{(a^2+y_0^2)^{\frac{3}{2}}}}, \quad \ldots \quad (70)$$

y_0 being the extreme half-breadth. When there is but one pair of foci, this becomes simply $\sqrt{a^2+y_0^2}$.

For the disturbance caused by a sphere, half immersed, equation (67) takes the following form,

$$Z = r \cdot \frac{2\cos^2\theta - 1}{3\cos^2\theta - 1}; \quad \text{(71)}$$

in which r is the horizontal distance from the centre of the sphere, and θ the angle that r makes with the axis of x; and the same value of Z is approximated to at distances from a disturbing solid of any figure which are very great compared with the dimensions of the solid.

The following examples are calculated for the oval neoïd of revolution LB, and for the cycnoïd of revolution L′B′, shown in fig. 3, the unit of measure being one tenth part of the distance from the axis OX to the nearest of the straight lines that are parallel to it,—also for a sphere of the radius 1.

	Oval.	Cycnoïd.	Sphere.
Half-length l	64	95	1
Extreme half-breadth y_0=greatest depth of immersion	26	31·6	1
Mean virtual depth Z_m	19·1	20·5	$\frac{2}{3}$
Virtual depth at ends Z_l	13·2	13·5	$\frac{1}{2}$
Virtual depth amidships Z_{y_0}	55·5	62·4	1

When *a wave of a given length* travels in water of unlimited depth, the virtual depth of disturbance is equal to the radius of a circle whose circumference is equal to the length of the wave. For *a wave of a given periodic time*, in water of unlimited depth, the virtual depth is equal to the height of a revolving pendulum which makes one revolution in the period of a wave. For *a wave travelling at a given speed*, under all circumstances whatsoever, the virtual depth is twice the height due to the speed; and conversely, *for a given virtual depth*, under all circumstances, the speed is that acquired during a fall through half the depth. (See Proceedings of the Royal Society, 16th April, 1868, page 345.) These laws are expressed as follows. Let W be the speed of advance of a wave in a horizontal direction perpendicular to the line of its crest, λ its length, T its period; then we have

in water of any depth, limited or unlimited,

$$W^2 = gZ; \quad \text{(72)}$$

and in water of unlimited depth,

$$T = 2\pi\sqrt{\frac{Z}{g}}; \quad \text{(72 A)}$$

$$\lambda = WT = 2\pi Z; \quad \text{(72 B)}$$

and therefore

$$W^2 = \frac{g\lambda}{2\pi}; \quad \text{(72 C)}$$

$$W = \frac{gT}{2\pi}. \quad \text{(72 D)}$$

§ 18. *Probable Laws of Wave-resistance.*—It has been proved by observation that a floating solid, such as a ship, is accompanied by waves, originating in the disturbances which it produces in the level of the water. None of those waves (at all events none whose energy is appreciable) travel faster than the floating solid. Some travel at the same speed, and some slower—each wave having its velocity in a direction normal to its crest regulated by its virtual depth, according to equation (72).

Those waves may be divided into three classes. The *first class*, whose properties were pointed out by Mr. SCOTT RUSSELL about twenty-five years ago, have a speed, and therefore a virtual depth, depending on the periodic time which elapses between the raising of a swell by the fore body and after body of the vessel respectively.

In the *second class* the virtual depth is regulated by the mean virtual depth of the whole longitudinal disturbance (68)—that is, by the mean depth of immersion of the vessel; the existence of these waves has been proved by observations of several actual vessels, some of which are described in a paper read to the British Association in 1868 (see the Reports for that year, p. 194; also the Transactions of the Institution of Naval Architects, 1868, p. 275; and the 'Engineer' for the 28th August and 30th October, 1868).

The waves which have been found by observation most distinctly to follow this law, are a pair of diverging waves which closely follow the stern of the vessel.

The *third class* of waves appear to depend on the several virtual depths of disturbance at various points in the neighbourhood of the vessel, especially at and near the bow. They diverge at various angles; and travelling into water in which the virtual depth increases, they become accelerated, so that their ridges are gradually curved forward. The general theory of this class of waves has been stated in the papers already referred to in connexion with the second class; but, so far as I know, they have not yet been subjected to exact observation, for which perfectly smooth water is necessary.

When a wave accompanies a disturbing body whose speed is greater than that of the wave, the direction of advance of the wave, which is perpendicular to its ridge-line, adjusts itself so as to make with the direction of advance of the vessel an angle whose cosine is the ratio borne by the speed of the wave to the speed of the ship; that is to say, let W be the speed of the wave, V that of the ship, α the angle of obliquity of the advance of the wave, then

$$\cos\alpha=\frac{W}{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

(see Transactions of the Institution of Naval Architects for 1864, vol. v. p. 321; also WATTS, RANKINE, NAPIER, and BARNES, on 'Shipbuilding,' p. 79). The effect of the divergence of a wave is to disperse, to distant parts of the water, a certain quantity of energy which is never restored to the vessel, and thus to cause a kind of resistance which may be called *wave-resistance.* It has been suggested, as a probable law of the rate at which a diverging wave disperses energy; that this rate is proportional to the *breadth of new wave* raised in a second; which breadth is equal to the speed of the vessel multiplied by the sine of the angle of obliquity of the wave; that is,

$$V \sin \alpha = \sqrt{V^2 - W^2}; \quad \ldots \ldots \ldots \quad (74)$$

and if this be correct, the resistance arising from the dispersion of energy by a set of waves of a given speed may be expressed as follows:—Let R′ be the propelling force which would be required in order to produce the disturbance constituting the wave-motion, if the whole of the energy of that motion were dispersed; then the actual propelling force required in order to restore the energy dispersed by those waves will be

$$R' \sin \alpha = R' . \sqrt{\left(1 - \frac{W^2}{V^2}\right)}. \quad \ldots \ldots \ldots \quad (75)$$

The total wave-resistance of a ship, according to this hypothesis, is the sum of a set of terms similar to the above expression, each term belonging to a different set of waves and containing its proper values of R′ and of W′.

Each value of R′ is probably proportional to the square of the speed of the ship, and to some function of her dimensions and of the position of that part of her where the set of waves in question originates, and may therefore be expressed in units of weight by $\frac{\omega V^2 \varrho}{2g}$, where ω is such a function, and ϱ the density of the water. Hence the total wave-resistance may be expressed as follows:

$$\Sigma . R' \sin \alpha = \frac{V^2 \varrho}{2g} . \Sigma \left\{ \omega \sqrt{\left(1 - \frac{W^2}{V^2}\right)} \right\}. \quad \ldots \ldots \quad (76)$$

For waves of the first class the value of W is that given by equation (72 D), the period T_1 being expressed as follows,

$$T_1 = \frac{f_1 l_1 + f_2 l_2}{V}; \quad \ldots \ldots \ldots \quad (77)$$

where l_1 and l_2 are the lengths of the fore body and after body respectively, and f_1 and f_2 two coefficients, depending on the forms of those bodies. From the practical results of the rules given by Mr. SCOTT RUSSELL, there seems to be reason to believe that those coefficients are sensibly equal to, or not very different from, the *coefficients of fineness*, found by dividing the displacement of the fore body and after body respectively by the area of midship section. The speed of waves of the first class is thus given by the following formula,

$$W_1 = \frac{g(f_1 l_1 + f_2 l_2)}{2\pi V}; \quad \ldots \ldots \ldots \quad (78)$$

and in order that such waves may not disperse energy by their divergence, it is necessary that W_1 should be equal to or greater than V; that is to say, that

$$f_1 l_1 + f_2 l_2 = \text{or} > \frac{2\pi V^2}{g}. \quad \ldots \ldots \ldots \quad (79)$$

It appears further, from results of practice, that it is advisable that the two terms of the left-hand member of this equation should be equal to each other; that is to say,

$$f_1 l_1 = f_2 l_2 = \text{or} > \frac{\pi V^2}{g}; \quad \ldots \ldots \ldots \quad (80)$$

and if we make $f_1=\frac{1}{2}$ and $f_2=\frac{3}{4}$, this becomes Mr. SCOTT RUSSELL'S rule for the least lengths of fore and after body suited to enable a ship to be driven economically at a given speed.

It is well known that in water that is shallow, compared with the length of a wave, waves of a given period are retarded according to a certain law (see AIRY on Tides and Waves). Hence the fact, which has often been observed, that a length which is sufficient for a given speed in deep water, becomes insufficient in shallow water—the waves of the first class becoming divergent, and the swell under the after body lagging behind, so as to make the stern of the vessel "squat," as it is called.

For waves of the second class, the value of W is given by equation (72), putting for Z the value given by equation (68)—that is, the mean depth of immersion D÷S. Hence we have

$$W^2=\sqrt{\frac{gD}{S}}; \quad \ldots \ldots \ldots \ldots \quad (81)$$

and this is probably unaltered in shallow water. The period of these waves is the same with that of the dipping, or vertical oscillation of the ship, whose value in deep water is

$$T_2=2\pi\sqrt{\frac{D}{gS}}. \quad \ldots \ldots \ldots \ldots \quad (82)$$

Waves of the third class are observed to have, as theory indicates, a great angle of obliquity at and near the bow of the vessel, gradually diminishing as they travel to more distant masses of water where the virtual depth is greater. Beyond this general agreement, their precise laws are not yet known, for want of a sufficient number of precise observations.

The general nature of the phenomena of wave-resistance, as indicated both by theory and by observation, are as follows. When either the speed of the vessel is so small, or her dimensions so great, as to make the ratio $\frac{W}{V}$ of the speed of each set of waves to that of the vessel greater than or equal to unity, in other words, to make the ratio $\sqrt{\left(1-\frac{W^2}{V^2}\right)}$ of the breadth of new wave raised per second to the speed of the ship nothing or imaginary, there is no wave-resistance, and the only resistances to be overcome in driving the ship at a uniform speed are that due to stiffness or viscosity, and that due to friction or "skin-resistance." The first of these increases simply as the speed; and at the velocities usual in navigation, it becomes almost inappreciable when compared with the resistance due to friction. At very low speeds it is the principal resistance. Its laws have been fully investigated by Mr. STOKES.

The resistance due to friction increases sensibly as the square of the speed. Some remarks on this kind of resistance will be added in the next section.

So soon as the ratio $\frac{W}{V}$ becomes less than unity for any set of waves, wave-resistance begins to be felt, and shows its nature by increasing more rapidly than the square of the speed; and its effects become more and more conspicuous as additional sets of waves come successively into operation as means of dispersing energy.

When the speed of the disturbing body becomes so great that, for all or for most of the sets of waves, the ratio $\frac{W}{V}$ becomes a very small fraction, the whole, or nearly the whole of the energy of disturbance is dispersed and wasted, and wave-resistance becomes the principal, or it may be the only appreciable resistance. In this extreme case it is possible to make a theoretical estimate of the amount of that resistance, as follows. The whole energy of disturbance is expressed, in absolute units, by

$$V^2 \varrho E_D,$$

a function of which values have been given in equations (47 A), (53), (60), &c.

The total dispersion of that quantity of energy, and its reproduction by the disturbing action of the solid, may be considered as taking place while the midship section M sweeps through a space equal to the displacement D of the solid—that is, while the solid advances through the distance $\frac{D}{M}$; and hence the propelling force required to overcome wave-resistance will probably have the following value, in *units of weight*,

$$\Sigma . R' = \frac{V^2 \varrho E_D M}{g D}; \quad \ldots \ldots \ldots \ldots \quad (83)$$

and the resistance will again increase as the square of the velocity.

The only solid *of continuous figure* on which experiments have been made suitable for comparison with this formula is the sphere. For that body, equation (60B) informs us, we have $\frac{2E_D}{D} = \frac{1}{2}$, and $\frac{E_D}{D} = \frac{1}{4}$; therefore the extreme wave-resistance is

$$\Sigma . R' = \frac{V^2 \varrho M}{4g}; \quad \ldots \ldots \ldots \ldots \quad (83\text{A})$$

that is to say, it is equal to *the weight of a column of liquid of half the height due to the speed, on a base equal to the midship section*,—a result which agrees very closely with experiment.

Since a propelling instrument which acts by the reaction of the water, as a paddle, a screw, an oar, or a jet, drives the particles of water astern, it tends to diminish the height of the crest of a wave, and to increase the depth of a trough or hollow;—in the former case diminishing, and in the latter increasing the energy of the wave, which partly goes to waste in the case of divergence; and hence it follows that it is favourable to economy of power that such a propelling instrument should act on the crest, rather than on the hollow of a wave. This fact is well known in practice.

The production of diverging waves is not prevented by totally submerging the disturbing body; but those waves are of less height at the surface of the water, the more deeply the body is covered. The virtual depth, and consequently the speed, of the waves of the second and third classes increases, and their angle of divergence diminishes, with increased submergence of the body; but the speed, and consequently the angle of divergence, of the waves of the first class is unaltered, because they depend on the time occupied by the solid in moving through a certain portion of its length.

§ 19. *Remarks on the Skin-resistance.*—It is well known through observation:—that the friction between a ship and the water acts by producing a great number of very small eddies in a thin layer of water close to the skin of the vessel, and also an advancing motion in that layer of water; that this *frictional layer* (as it may be called) is of insensible thickness at the cutwater, and gradually increases in thickness towards the stern, by the communication of the combined whirling and progressive motion to successive streams of particles; and that, finally, the various elementary streams of which the frictional layer is composed, uniting at the stern of the ship, form her *wake*—that is, a steady or nearly steady current, full of small eddies, which follows the ship, but at a speed relatively to still water which is less than the speed of the ship.

The central stream of the wake has the greatest velocity ahead; and other parts of it have velocities diminishing from the centre towards the circumference. If the friction between the water and a given area of the skin of the ship is equal to that of an equal area of one layer of water upon another at a given velocity, the *mean* forward velocity of the whole wake relatively to still water, and its mean backward velocity relatively to the ship, are each of them equal to one half of her speed.

The effect of discontinuity of form, as when the figure of the vessel presents angles to the water, is to produce eddies which are dragged along with the ship, and thus to add to the wake; and hence the resistance arising from discontinuity of form is analogous in its laws to that arising from friction; and both those forces are comprehended under the name of *eddy-resistance.* Bodies of discontinuous forms, however, are foreign to the subject of this paper.

Let V, as before, be the velocity of the ship; let W′ denote the mean velocity of the wake, and C its area of cross section, both taken at a distance astern of the vessel sufficient for the wake to have become a steady forward current. Let R be the amount of the skin-resistance in units of weight, and ϱ the density of water. Then the mass of water added to the wake in each second is $\varrho C(V-W')$; and the velocity impressed on that mass by the force R is W′; whence we have the following equation,

$$R=\frac{1}{g}\cdot\varrho C(VW'-W'^2); \quad \ldots \ldots \ldots \quad (84)$$

and if the mean velocity of the wake is half the velocity of the ship, that equation becomes

$$R=\frac{\varrho CV^2}{4g}. \quad \ldots \ldots \ldots \quad (84\text{ A})$$

It is obvious from equation (84) that, for a given amount of skin-resistance, the wake has the *least possible sectional area* when its mean speed is half that of the ship.

The work done by the ship on the water per second in producing the wake is RV; the actual energy of the current of the wake is increased in each second by the amount

$$\frac{\varrho C}{g}\cdot(V-W')\frac{W'^2}{2};$$

and the difference between those quantities—that is,

$$RV-\frac{\varrho C}{g}(V-W')\frac{W'^2}{2}=\frac{\varrho C}{g}(V-W')\left(VW'-\frac{W'^2}{2}\right), \quad . \quad . \quad . \quad . \quad (84\text{ B})$$

is the energy added to that of the eddies in each second. If, as before, we have $W'=\frac{1}{2}V$, the preceding equation takes the following value,

$$RV-\frac{\varrho CV^3}{16g}=\frac{3\varrho CV^3}{16g}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84\text{ C})$$

so that one fourth of the work of friction is expended in producing the current in the wake, and the other three fourths in producing eddies.

If the velocities V and W′ of the ship and her wake, and the amount of eddy-resistance R, are given, the sectional area C of the wake may be calculated from equation (76).

The elementary streams of which the wake is composed move astern relatively to the ship with a velocity less than that of an undisturbed current in the ratio expressed by $\frac{V-W'}{V}$; and hence they occupy a transverse area greater than they would do in the undisturbed state in the ratio expressed by

$$\frac{V}{V-W'}=1+\frac{W'}{V-W'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

which, when $W'=\frac{1}{2}V$, becomes $=2$. This causes a certain modification in the forms of the stream-lines outside the wake, which might be represented by taking for the surface of an imaginary disturbing solid a surface midway between the skin of the vessel and the outer surface of the frictional layer, followed by an indefinitely long cylindrical tail of one half of the sectional area of the wake; but the detailed investigation of this will not now be entered on.

Mr. FROUDE a few years ago pointed out that the most perfect propeller for driving a ship against skin-resistance, would be one which should act solely on the particles of the wake, driving them astern so as just to take away their forward velocity and no more. The velocity of such a propeller relatively to the ship would be equal and opposite to her speed V; and the energy expended in working it would be simply RV, equal to the work done by the ship, through friction, on the water. It would thus be a propeller free from "slip" and free from waste of power. It would stop the following current in the wake, and would at the same time impress on the water an additional quantity of energy in the form of eddy-motion, equal to the energy taken away in stopping the current; so that the total energy impressed on the water in each second would be the same as before.

It would preserve to the stream-lines the shape which they would have in the absence of friction.

A propeller of the most efficient kind possible, producing the same forward thrust R, by acting on previously undisturbed water so as to impress a backward velocity W″ on a current of the sectional area B, would move it astern relatively to the ship with the velocity $V+W''$; and besides expending in each second the quantity of energy RV in driving the ship ahead, it would expend the additional quantity RW'' in driving the water astern. The relation between the sectional area and the velocity of the current produced by such a propeller is given by the following equation,

$$R=\frac{\varrho B}{g}(V+W'')W'', \quad \ldots \ldots \ldots \quad (86)$$

because $\varrho B(V+W'')$ is the mass of water acted on in each second, and W'' the velocity impressed on it. The *counter-efficiency*, being the ratio in which the total work done exceeds the useful work, is

$$1+\frac{W''}{V}. \quad \ldots \ldots \ldots \quad (86\text{ A})$$

In previous writings* it has been shown that the amount of skin-resistance is probably expressed by a formula of the following kind,

$$R=\frac{\varkappa\varrho V^2}{2g}\cdot\iint q^3 d\omega; \quad \ldots \ldots \ldots \quad (87)$$

in which $d\omega$ is the area of an elementary portion of the skin of the ship, $q=\sqrt{(u^2+v^2+w^2)}$ the ratio borne by the velocity with which the particles of water glide over that elementary area, to the velocity of the ship (V), ϱ the density of water, and $\varkappa$ a coefficient of friction, whose value, as deduced from the performance of actual ships, is about ·0036 or ·004 for a clean surface of painted iron.

The integral $\iint q^3 d\omega$ is called the *augmented surface*; and the ratio

$$\frac{\iint q^3 d\omega}{\int G dx} \quad \ldots \ldots \ldots \quad (87\text{ A})$$

is called the *coefficient of augmentation*. The denominator, $\int G dx$, is what may be called the girth-integral, G denoting the immersed girth of a given cross section of the vessel. The augmented surface and coefficient of augmentation can be calculated for any particular stream-line surface by drawing it, constructing such a diagram as that shown in fig. 5, and finding approximate values of the definite integrals by SIMPSON'S Rules; but to give exact general symbolic expressions for them involves difficulties which have not yet been overcome.

The following are particular cases in which exact expressions have been found:—

Indefinitely deep circular cylinder of radius l, $q=2\sin\theta$;
Augmented surface per unit of depth, $21\frac{1}{3}l$;
Coefficient of augmentation, $\frac{16}{3}=5\cdot 3$ nearly.
Sphere of radius l, $q=\frac{3}{2}\sin\theta$;

* Philosophical Transactions 1863, p. 134; 1864, p. 384; Civil Engineer and Architect's Journal, October 1861; Transactions of the Institution of Naval Architects for 1864, vol. v. p. 322; Shipbuilding, Theoretical and Practical.

Augmented surface, $\frac{81}{32}\pi^2 l^2 = 25 l^2$ nearly;

Girth-integral, $\int G dx = \pi^2 l^2 = 9{\cdot}87 l^2$ nearly;

Coefficient of augmentation, $\frac{81}{32} = 2{\cdot}531$ nearly.

In each case θ denotes the angle made by a given radius with the direction of motion.

For a sphere half-immersed the augmented surface and girth-integral have respectively half the values given above.

For an approximately trochoidal riband of uniform breadth, it has been elsewhere shown (Philosophical Transactions, 1863, p. 134) that the coefficient of augmentation is very nearly $1+4\sin^2\beta + \sin^4\beta$, β being the angle of greatest obliquity of the riband to the direction of motion.

With a view to the calculation of the augmented surface by numerical definite integration in particular cases, the following values of the elementary surface $d\omega$ and of its first integral are given. As to the function χ, see § 3.

General case:

$$d\omega = \sqrt{\{dy^2dz^2 + dz^2dx^2 + dx^2dy^2\}}. \quad \dots \quad (88)$$

Cylindrical surface of indefinite depth; $d\omega$ per unit of depth $= \frac{qdx}{u}$. . . . . . (88 A)

Surface of revolution, *half-immersed*; $\int d\omega$ for a zone or belt measuring dx lengthwise $= \frac{\pi yqdx}{u}$. . . . . (88 B)

§ 20. *General Remarks.*—The dynamical investigations contained in this chapter are partly certain and exact, partly approximate, and partly conjectural. The results arrived at in §§ 12 to 15 as to momentum and energy of current and of disturbance, are all certain and exact when applied to the case of a solid body of any figure past which a fluid can glide continuously, immersed in an unlimited mass of liquid, and approximate when applied to cases such as those described in § 16, in which these conditions are approximately fulfilled. The results as to virtual depth of disturbance, and as to speed of waves, in § 17, are partly exact, and partly approximate. The probable laws of wave-resistance and of skin-resistance, in §§ 18 and 19, are partly conjectural, and require the aid of much additional experimental research to test and verify them, and to make them definite; but still they have already to a certain extent been verified by observations of the performance of ships. The whole body of results, whether certain or conjectural, are set forth in the hope that they may prove useful in deducing general principles from the data of experiment and observation, and in suggesting plans for further research.

SUMMARY OF THE CONTENTS.

CONTENTS OF SUPPLEMENT.

Supplement to a paper on the Mathematical Theory of Stream-lines. *By* WILLIAM JOHN MACQUORN RANKINE, *C.E., LL.D., F.R.SS. Lond. & Edin.*

Received January 8,—Read February 10, 1870.

I. *Addendum to* § 16.—*Points of no Disturbance of Pressure.* Mr. BERTHON'S *Log.* —The points in the surface of the disturbing solid, and elsewhere, at which there is no disturbance of pressure, are given by the equation

$$q^2=u^2+v^2+w^2=1. \qquad (a)$$

Such points can be found graphically for a given stream-line surface, by constructing a diagram such as fig. 5, and finding by trial the points for which AC=AB.

At the surface of a sphere it is easily shown that we have

$$q=\tfrac{3}{2}\sin\theta; \qquad (b)$$

in which θ is the angle made by a radius of the sphere with the direction of motion. Hence, on the surface of a sphere, the points of no disturbance of pressure are contained in the circle given by the equation

$$\theta=\sin^{-1}\tfrac{2}{3}=41^\circ\ 59'\ \text{nearly}.$$

In February 1850 there was communicated to the Royal Society a paper by the Reverend E. L. BERTHON, describing an instrument invented by him, called a "hydrostatic log;" and a more detailed account of that invention was read by Mr. VAUGHAN PENDRED to the Society of Engineers on the 6th of December 1869. One part of that instrument consists of a vertical cylindrical tube, with a closed flat bottom, and having, in the front part of the cylindrical surface, near the bottom, a small hole, whose angular position, relatively to the direction in which the tube is moved through the water, is so adjusted that the pressure of the water outside produces no disturbance of the level of the column inside the tube. Mr. BERTHON ascertained solely by experiment the "zero-angle" or "neutral angle," as it has been called, and found it to be 41° 30′—a result with which the theoretical value for a sphere agrees almost exactly. That agreement shows that the disturbance in the water caused by the short vertical flat-bottomed cylinder employed by Mr. BERTHON was sensibly identical with that produced by a sphere, and also that, from the foremost points of the tube, as far round each way as the zero-angle, the disturbance of pressure was not sensibly affected by wave-motion, viscosity, or friction.

II. *Addendum to* § 17.—*Interference of Waves.* It was suggested to me by Mr. WILLIAM FROUDE, in a letter dated the 11th November, 1869, that one of the circumstances in the figure of a vessel on which the smallness of wave-resistance depends, is the interference of waves originating at different parts of the vessel's surface, so as wholly or partially to neutralize each other. Suppose, for example, that at a certain point (which may be denoted by A), at or near the bow of the ship, there is a disturbance producing a wave-ridge. This first ridge is followed by a series of other wave-ridges, at distances apart depending on the period and virtual depth of the original disturbance, and diverging at an angle depending on the ratio borne by the speed of the waves to the speed of the ship.

Mr. FROUDE remarks, as an observed fact, that the *second* wave-ridge of the series is that which appears to carry away the most energy. This wave-ridge is in contact with the side of the vessel at a point which we may call B, at a distance astern of A depending on the dimensions and position of the waves. Suppose, now, that the surface of the vessel is so shaped that the disturbance impressed by it on the water has a tendency to produce a *wave-trough* at B; this disturbance will, to a certain extent, neutralize, by interference, the disturbance originating at A; that is, to use Mr. FROUDE'S words, the wave-troughs originating at B will "swallow" the second and following ridges of the series of waves originating at A, leaving unaltered the first wave-ridge only of that series—thus diminishing the quantity of energy which is carried away by diverging waves. It is obvious that the greatest effect of the interference of two given series of waves can be realized at one particular speed of the ship only, because of the influence of the speed of the ship on the positions of the waves; and this may account for the diminutions of the rate of increase of resistance with speed which occur at certain particular velocities of a given vessel. As regards this and many other questions of the resistance of vessels and of the motions which they impress on the water, a great advancement of knowledge is to be expected from the publication in detail of the results of the experiments on which Mr. FROUDE has long been engaged.

FIG 2.

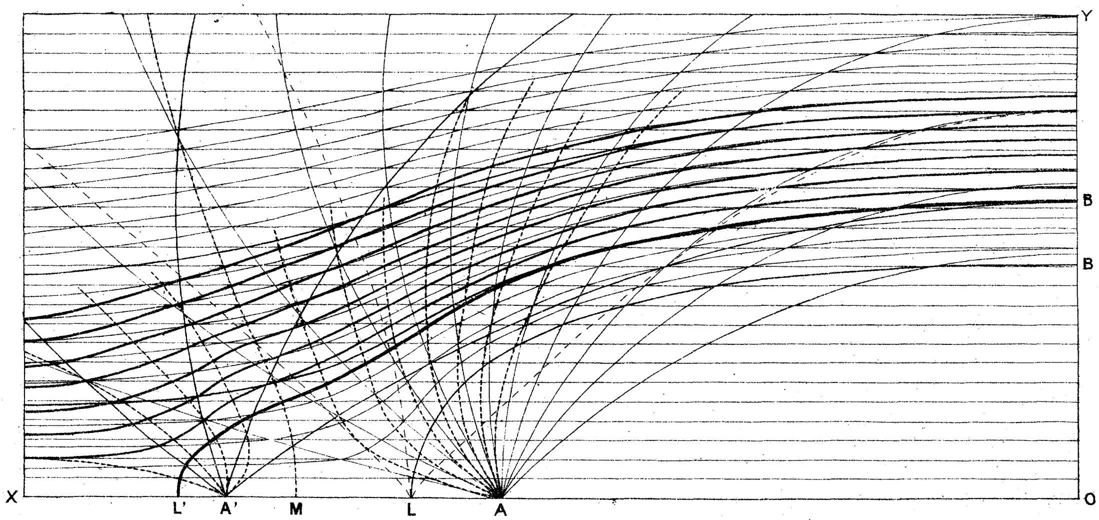


FIG 3.

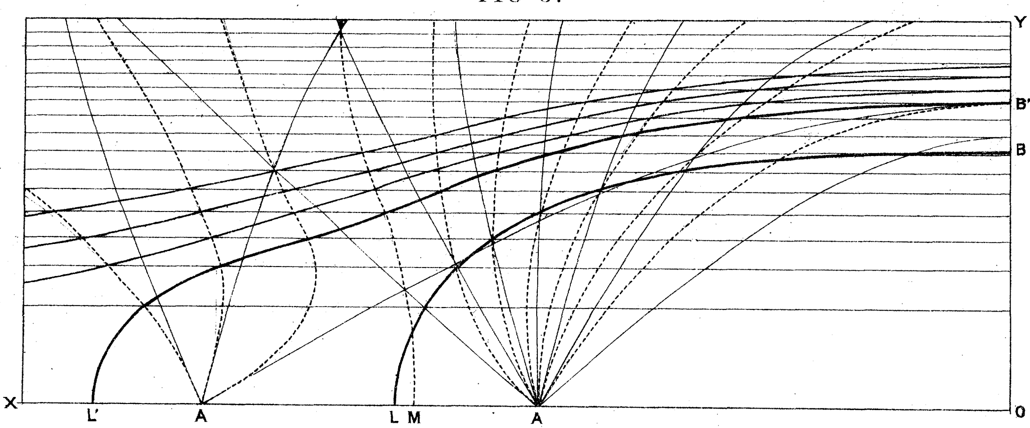


FIG 1.

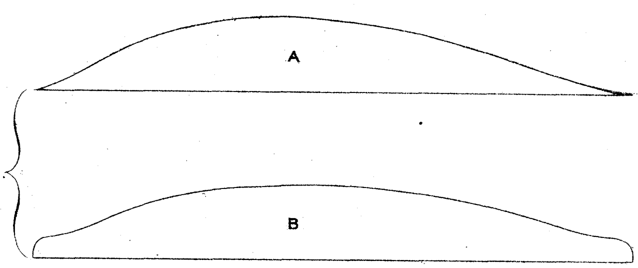
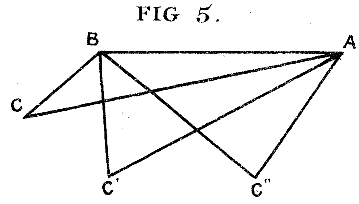


FIG 5.



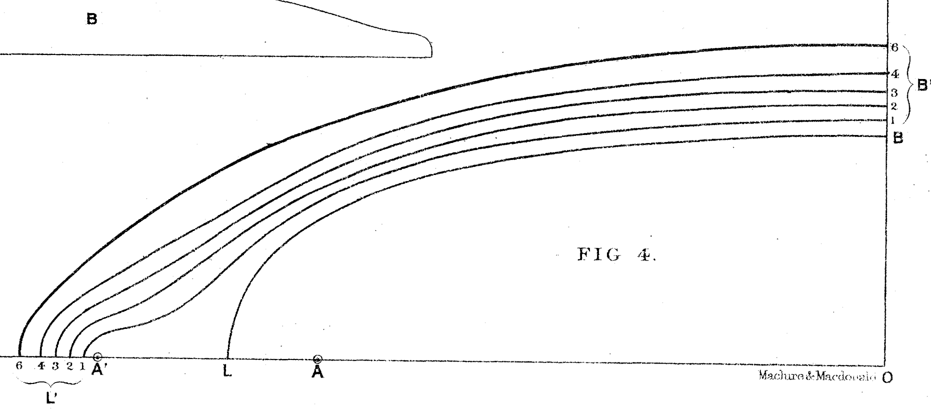


FIG 4.

Drawn by W.J.M.R.

Maclure & Macdonald